SHAPE 4

Reflector Design Software

Mathematical Description

Finite Light Source

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Rotational symmetry

1. Description of a problem

The synthesis of reflector includes the process of construction of zones that provide a requested light distribution. This process needs calculation of light values (illuminance or luminous intensity) from the zones, i.e. the direct problem is called during the synthesis on each step of solution of inverse problem.

2. Remote (infinite) distances

Direct light

Luminous intensity on the direction $\hat{a}_0$

$$I(\hat{a}_0) = \iint L ds = \int_{y_1}^{y_2} L(x_2 - x_1) dy \tag{1}$$

where

L - is the luminance of a flashed area ds of light source when viewed from the point P along the direction $\hat{a}_0$

x_2, x_1 - coordinates of the light source projection on the plane perpendicular to the direction $\hat{a}_0$

Reflected light

Luminous intensity on the direction $\hat{a}_0$

$$I(\hat{a}_0) = \rho \iint L \cos\theta ds \tag{2}$$

where

L - is the luminance of a flashed area ds of reflector when viewed along the direction $\hat{a}_0$

$$\cos\theta = (\hat{a}_0, \hat{n})$$

where

the normal vector is

$$\hat{n} = \begin{bmatrix} \sin\delta \sin\psi & \sin\delta \cos\psi & \cos\delta \end{bmatrix}^T$$

the viewing vector in the axially symmetric reflectors becomes

$$\hat{a}_0 = \begin{bmatrix} \sin\alpha\sin\beta & \sin\alpha\cos\beta & \cos\alpha \end{bmatrix}^T$$

or from $\beta = 0$

$$\hat{a}_0 = \begin{bmatrix} 0 & \sin\alpha & \cos\alpha \end{bmatrix}^T$$

In the axially symmetric becomes

$$ds = \frac{r^2 \sin\varphi}{\cos(\varphi - \delta)} d\varphi d\psi$$

We obtain from Eq. (2):

$$I(\alpha) = \rho \iint L(\cos\alpha\cos\delta + \sin\alpha\sin\delta\cos\psi) \frac{r^2 \sin\varphi}{\cos(\varphi - \delta)} d\varphi d\psi \tag{3}$$

From $L = L_0$

$$I(\alpha) = \rho L_0 \int_{\varphi_s}^{\varphi_e} \frac{r^2 \sin\varphi}{\cos(\varphi - \delta)} \{T(\psi_{up}) - T(\psi_1)\}\, d\varphi \tag{4}$$

where the function

$$T(\varphi) = \cos\alpha\cos\delta\,\psi + \sin\alpha\sin\delta\sin\psi,$$

where integration limits φ_s, φ_e and φ_1, φ_{up} can be found by inverse-ray method.

The search of flashed area (image) is being carried out by making fixed steps with respect to φ and ψ - inverse-ray methods.

Significant reducing of calculation time can be obtained by defining envelope sources - e.g. for a coaxial cylinder of uniform luminance, the inverse-ray tracing procedure enables to find analytical expressions for image boundaries $\psi(\varphi)$ along the parallel φ = const.

$$\tan(\beta_i/2) = \frac{\bar{k}_i}{1 + \sqrt{1 - \bar{k}_1 \bar{k}_2}} \tag{5}$$

where

$$\bar{k}_1 = \sin(2\delta - \alpha)\tan\psi_r / \sin\alpha$$
$$\bar{k}_2 = \sin(2\delta + \alpha)\tan\psi_r / \sin\alpha$$

$$\tan\psi_r = \frac{r_c}{\sqrt{y_0^2 - r_c^2}}, \quad y_0 = r_c \sin\varphi$$

r_c - radius of cylinder

3. Finite distances

Direct light

Horizontal illuminance from a light source is

$$E_h = \frac{L_0}{2} \iint_{\Omega} \sin 2\alpha \, d\alpha \, d\beta \qquad (6)$$

where $\Omega(\alpha,\beta)$ specifies the limits of integration related to the dimensions of a domain percepted as bright from the viewing point.

Algorithm of computing the illuminance consists of the following stages:
1) formation of inverse-ray vector:

$$\hat{a}_0 = \begin{bmatrix} \cos\alpha & \sin\alpha\sin\beta & \sin\alpha\cos\beta \end{bmatrix}^T$$

2) calculation of coordinates of a point where the ray intersects a source surface;
3) calculation of integrals of illuminance Eq.(6)

Reflected light

In accordance with the known expression for the illuminance at the point P, we have

$$E = \rho \iint \frac{L\cos\theta_1 \cos\theta_2}{R^2} ds \qquad (7)$$

where

L - is the luminance of a flashed area ds of reflector when viewed from the point P along the direction $\hat{a}_0$
R - is the distance from the point P to the element ds on the reflector

$$\cos\theta_1 = (\hat{a}_0, \hat{n}) \quad \cos\theta_2 = (\hat{a}_0, \hat{n}_p)$$

where

$\hat{n}_p$ - is the unit normal vector to the point P on the working plane

If the point P determinated by vector $\vec{P} = [x_0 \quad y_0 \quad z_0]^T$ and the point on the reflector $\vec{M} = [x \quad y \quad z]^T$ then

$$\hat{a}_0 = \frac{(\vec{M} - \vec{P})}{\|\vec{M} - \vec{P}\|} = \frac{[x - x_0 \quad y - y_0 \quad z - z_0]^T}{R} \tag{8}$$

For axially rotational reflector and uniform luminance of light source ($L = L_0$) and $\hat{n}_p$ = $[0 \;\; 0 \;\; 1]^T$ we get from Eq.(7)

$$E_0(P) = \rho L_0 \int_{\varphi_s}^{\varphi_e} \int_{\psi_1}^{\psi_{up}} \frac{(\hat{a}_0, \hat{n})(z - z_0)}{R^3} \frac{r^2 \sin\varphi}{\cos(\varphi - \delta)} d\varphi d\psi \tag{9}$$

where
$\hat{a}_0$ - is calculated from Eq.(8)
$\hat{n} = [\sin\delta \sin\psi \quad \sin\delta \cos\psi \quad \cos\delta]^T$
$x = r\sin\varphi\sin\psi$
$y = r\sin\varphi\cos\psi$
$z = r\cos\psi$

4. The rules of constructing a surface

A surface is being assembled from the finite number of zones, the contours of which are described by equations of the second order. To reproduce the prescribed illuminance diagram, we use elliptic or hyperbolic profiles; and in order to fit the prescribed candlepower curve, we use parabolic profile. The construction of total profile curve is governed by the scheme of ray tracing and orientation of illuminated area $[x_{beg}, x_{fin}]$ (see Table 1).

Table 1

RT scheme	*Zone type*	
	ID case	*LID case*
Non-crossed	Elliptic or hyperbolic	Parabolic
Crossed	Elliptic	Parabolic

Of course, these limitations concerning the shape of curves decrease the number of possible solutions, but still can not be the cause of failure in designing specific type of reflector. We have to select the curve class during inverse calculation, when the ambiquity of solution is principally inherent.

The asembling of prescribed candlepower curve by summing zonal curves is governed by the following rules: the zonal curve ΔE_i (or ΔI_i) should not alter the value of a summerized curve in the preceding point (xi-1). The number of zones is equal to the number of interpolation points in candlepower curve being assigned on the axis OX. The zones have smooth conjugation up to the 2nd order, i.e. they expose one (continuous) normal at the point of function.

4.1. Elliptic zone

Elliptic zone is used either in crossed, or in direct scheme of ray tracing. But, in the latter case elliptic zone has limited abilities in creating minimal illuminance increment.

In every step of calculation the position of focus F on some basic line is assumed as a zonal parameter. The focus position depends on location of preceding profile point and its normal, as well as on the next point of interpolation. In essence, a part of ray length from the support point to the source contour is considered to be a parameter.

If (r_0,ϕ_0) are the coordinates of the support point in the coordinate system associated with ellips which has its focus at the varied point F and $2c_f = \|x_i\ F\|$, then the eccentricity e can be found by solving a quadratic equation $(0 < e < 1)$:

$$e = (-B \pm \sqrt{D}) / A \qquad (10)$$

where

$$\begin{aligned} A &= r_0 \cos\phi_0 + c_f , \\ B &= 0.5\, r_0, \\ D &= B^2 - A\, C, \\ C &= -c_f; \end{aligned} \qquad (11)$$

Other parameters are as follows:

- parameter

$$p = r_0 (1 + e \cos\phi_0)$$

- are semiaxes $(a > b)$

$$a = p (1 - e^2)$$

$$b = \sqrt{a^2 - c^2}$$

- final angle

$$\phi_1 = \phi_0 + 2\alpha_v,$$

where α_v is an angle size of a source being observed from the focal point F (α_v has an algebraic meaning);

$r_1 = p/(1 + e \cos \phi_1)$ is a final radius-vector.

4.2. Hyperbolic zone

This profile is used as additional in the direct ray-tracing scheme, when elliptic zone gives too great intensities. In the proposed algorithm the choice is being carried out among the branches of hyperbola - the near (to the source) and the remote - it depends on required values of illuminance increment.

For the remote branch (being convex at the direction towards a source) the coefficients of quadratic equation [similar to Eqs. (10) and (11)] have the following form ($e > 1$):

$$A = r_0 \cos\phi_0 + c_f, \quad (12)$$
$$B = -0.5\, r_0,$$
$$C = c_f;$$

- parameter

$$p = r_0 (1 + e \cos\phi_0)$$

- are semiaxes (where $c^2 = a^2 + b^2$) (13)

$$a = p\,(e^2 - 1),$$

$$b = \sqrt{c^2 - a^2}$$

4.3. Parabolic zone

This profile is used to fit the required candlepower curve [1].
We have ($e = 1$)

$$p = r_0 (1 + \cos\phi_0) \quad (14)$$

$$r_1 = p/(1 + \cos\phi_1)$$

The ray tracing scheme can be crossed and direct (see Table 1 and [1]).

4.4. Constructing profile curve of a zone

Equation of a profile curve in the general coordinate system.

In local coordinate system the equation of a profile curve is as follows:

$$\alpha^2 x_1^2 \pm \beta^2 y_1^2 = 1 \quad (15)$$

where

$\alpha^2 = 1/a^2 \qquad \beta^2 = 1/b^2$

(the upper sign refers to ellipse, the lower, to hyperbola).

Equation for parabola

$$y_1 = 0.5(1 - x_1^2 / p) \tag{16}$$

At every i-th step of calculation we need to determine illuminance or candlepower in every subsequent (i+1,i+2,...) interpolating points in order to provide the required value of Ereq.

For this purpose we have to translate Eqs.(15) and (16) in the general coordinate system by:

1) rotation of coordinate system

$$\begin{bmatrix} x_l \\ y_l \end{bmatrix} = \begin{bmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

2) translation of origin of coordinates

$X = x - sx$

$Y = y - sy$

Algebraic equation of the second order curve in Cartesian coordinate system has a classical form:

$$a_{11}\, x^2 + 2\, a_{12}\, x\, y + a_{22}\, y^2 + 2\, a_1\, x + 2\, a_2\, y + a = 0 \tag{17}$$

For ellipse and hyperbola the coefficients of Eq.(17) are as follows:

$$a_{11} = 0.5\,((\alpha^2 \pm \beta^2) + (\alpha^2 \mp \beta^2)\cos 2v) \tag{18}$$

$a_{12} = -0.5\,(\alpha^2 \pm \beta^2)\sin 2v$

$a_{22} = 0.5\,((\alpha^2 \pm \beta^2) - (\alpha^2 \mp \beta^2)\cos 2v)$

$a_1 = -(a_{11}\, sx + a_{12}\, sy)$

$a_2 = -(a_{22}\, sx + a_{12}\, sx)$

$a = a_{11}\, sx^2 + a_{22}\, sy^2 + 2\, a_{12}\, sx\, sy - 1$

where v is an inclination angle between the ellipse and the vertical axis OY; (sx,sy) are the focus coordinates in the general system.

For parabola, the translation of the coordinate system gives:

$$a_{11} = \cos^2\alpha \tag{19}$$

$$a_{12} = -\sin\alpha \cos\alpha$$

$$a_{22} = \sin^2\alpha$$

$$a_1 = p \sin\alpha + sy \sin\alpha \cos\alpha - sx \cos^2\alpha$$

$$a_2 = p \cos\alpha + sx \sin\alpha \cos\alpha - sy \sin^2\alpha$$

$$a = (sx \cos\alpha - sy \sin\alpha)^2 - 2 p (sx \sin\alpha + sy \cos\alpha) - p^2$$

where α is a current angle, where the zone illuminates.

Polar form.

By substituting $x = r \sin\phi$, $y = r \cos\phi$, we translate Eq.(19) into polar form:

$$F(\phi) = A_{11} r^2 + 2 A_{12} r + A_{22} = 0 \tag{20}$$

where

$$A_{11} = a_{11} \sin^2\phi + a_{12} \sin2\phi + a_{22} \cos^2\phi \tag{21}$$

$$A_{12} = a_1 \sin\phi + a_2 \cos\phi$$

$$A_{22} = a$$

Equation (20) sets the function $r = r(\phi)$ and ray-tracing function that is necessary for calculating flashed area of profile curve

$$\alpha(\phi) = \phi + 2\mathrm{arctg}\left(0.5\frac{\delta A_{11}/\delta\phi r + 2\delta A_{12}/\delta\phi}{A_{11}r + A_{12}}\right) \tag{22}$$

where

$$\delta A_{11}/\delta\phi = (a_{11} - a_{22}) \sin2\phi + 2a_{12} \cos2\phi$$

$$\delta A_{12}/\delta\phi = a_{12} \cos\phi - a_2 \sin\phi$$

5. Flashed area equation for cylindrical infinite source

In case of cylinder, there are explicit formulas for calculating image size in cross section and observing it from infinity. When passing to finite distances these formulas have to be corrected. Assuming rotationally-symmetrical system, we define a movable observation point of a mirror element Y_0 by two coordinates: shift - Y_v, and an angle of rotation - β.
To obtain formulas of calculation we have to know just X,Y- coordinates of inverse ray $\mathbf{a_0}$:

$$a_{0X} = Y_v * \sin(\beta) / a,$$
$$a_{0Y} = (Y_v - Y_0) * \cos(\beta) / a. \tag{23}$$

A reflected ray is tangent to cylinder in condition that

$$Y_0^{\ 2} \cdot \xi^2 = r_c(\xi^2 + \eta^2) \ , \tag{24}$$

where ξ and η are X,Y-components of reflected (inverse) ray. Taking into account Eq.(23), we obtain from Eq.(24) a quadratic equation with respect to $t = \tan(0.5\beta)$

$$t^2 k_2 - 2\,t\,m + k_1 = 0 \ , \tag{25}$$

where

$$k_1 = \frac{\sin(2\delta - \alpha)}{\sin\alpha} \tan\psi_r \ ,$$
$$k_2 = \frac{\sin(2\delta - \alpha)}{\sin\alpha} \tan\psi_r \ ,$$
$$\tan\psi_r = \frac{R_c}{\sqrt{(Y_0^{\ 2} - R_c^{\ 2}}} \ ,$$
$$m = \frac{Y_v}{(Y_v - Y_0)} \ .$$

So, a correcting factor m is included to equation that describes image boundaries in cross section. Obviously, $m \to 1, if\ Y_v \to \infty$.

6. Full flux of light source

Formulas considered below are used for definition of flux from the light source.

Cylinder

$$F = 2F_{disk} + 2\pi^2 rlL$$

where

F_{disk} - flux from disk butt-end
r - radius of cylinder
l - length of cylinder
L - luminance of source

Sphere

$$F = 4\pi^2 r^2 L$$

where

r - radius of sphere
L - luminance of source

Ellipsoid

Let a - semiaxis perpendicular to reflector axis and b - semiaxis along reflector axis.

If b > a, the flux

$$F = 2\ \pi^2 abL\ \left\{\sqrt{1-e^2} + \arcsin(e)/e\right\}$$

where

$$e = \frac{\sqrt{b^2 - a^2}}{b}$$

If b < a, the flux

$$F = 2\ \pi^2 a^2 L\left\{1 + \left(1-e^2\right)\operatorname{arsh}\left(e/\sqrt{1-e^2}\right)/e\right\}$$

where

$$e = \frac{\sqrt{a^2 - b^2}}{a}$$

Disk

Flux in hemisphere

$$F = \pi^2 r^2 L$$

where

r - radius of disk
L - luminance of source

Torus

Flux from torus light source is calculated by numerical method as an integral of illuminance distribution on a sphere wrapping the torus.

Translational symmetry

7. Description of a problem

The formulated problem is considered as plane if the light tubes are formed by radial lines originating from the source center.

8. Remote (infinite) distances

In this case the horizontal illuminance at arbitrary point on the axis OX is

$$E = dF/dS = I\, d\alpha/(l\, dx) \qquad (26)$$

where dF is the luminous flux within the tube (α, α+dα); l is the length of a source; I_α is the candlepower of a light source.

Assuming x = H tgα, we obtain

$$E = (I_0/H) \cos^2\alpha \qquad (27)$$

where I_0 is a specific (per unit of length) candlepower of a light source; H is the elevation of light centre above working plane. Hence E_h is proportional to $\cos^2\alpha$ (not to $\cos^3\alpha$ as in spatial case).

For cylindric source with the radius R and constant luminance L_0 the candlepower is

$$I = 2\, R\, L_0\, l \qquad (28)$$

For a surface area assigned in polar coordinates (r_1,ϕ_1) and (r_2,ϕ_2) that illuminates at the direction, the increment of candlepower is equal to [1]

$$I_\alpha = L_0\, l\, \delta h = L_0\, l\, (r_2 \sin(\phi_2-\alpha) - r_1 \sin(\phi_1-\alpha)) \qquad (29)$$

This is the basic equation which serves for searching the parameters of a zone with known geometry that fits the required increment of candlepower at given direction.

9. Finite distances

For finite distances, in accordance with Wiener's theorem [1] we have

$$E = l\int_{\alpha_1}^{\alpha_2} L\cos\alpha d\alpha \tag{30}$$

where L is the luminance of rays within the ray cone with a vertex at the observed point x.

When the luminance of a light source is constant $L = L_0$, we have

$$E_h = L_0\, l\,(\sin\alpha_2 - \sin\alpha_1) \tag{31}$$

Equation (31) serves for searching zone parameters that fit the required increment of illuminance at given point on the axis OX. Following the recursion, we consider the point (r_1,ϕ_1) of a surface as being assigned, and after the next step find the point (r_2,ϕ_2) that fits the relation

$$\Delta E = \Delta E_{req} \tag{32}$$

10. The rules of constructing a surface

A surface is being assembled from the finite number of zones, the contours of which are described by equations of the second order. To reproduce the prescribed illuminance diagram [Eqs. (31) and (32)], we use elliptic or hyperbolic profiles; and in order to fit the prescribed candlepower curve [Eq.(28)], we use parabolic profile. The construction of total profile curve is governed by the scheme of ray tracing and orientation of illuminated area $[x_{beg},x_{fin}]$ (see Table 2).

Table 2

RT scheme	*Zone type*	
	ID case	*LID case*
Non-crossed	Elliptic or hyperbolic	Parabolic
Crossed	Elliptic	Parabolic

Of course, these limitations concerning the shape of curves decrease the number of possible solutions, but still can not be the cause of failure in designing specific type of reflector. We have to select the curve class during inverse calculation, when the ambiquity of solution is principally inherent.

The asembling of prescribed candlepower curve by summing zonal curves is governed by the following rules (see [1]): the zonal curve ΔE_i (or ΔI_i) should not alter the value of a summarised curve in the preceding point (x_{i-1}). The number of zones is equal to the number of interpolation points in candlepower curve being assigned on the axis OX. The zones have smooth conjugation up to the 2nd order, i.e. they expose one (continuous) normal at the point of function.

10.1. Elliptic zone

Elliptic zone is used either in crossed, or in direct scheme of ray tracing. But, in the latter case elliptic zone has limited abilities in creating minimal illuminance increment.

In every step of calculation the position of focus F on some basic line is assumed as a zonal parameter. The focus position depends on location of preceding profile point and its normal, as well as on the next point of interpolation. In essence, a part of ray length from the support point to the source contour is considered to be a parameter.

If (r_0,ϕ_0) are the coordinates of the support point in the coordinate system associated with ellips which has its focus at the varied point F and $2c_f = \|x_i\ F\|$, then the eccentricity e can be found by solving a quadratic equation ($0 < e < 1$):

$$e = (-B \pm \sqrt{D}) / A \qquad (33)$$

where

$$A = r_0 \cos\phi_0 + c_f , \qquad (34)$$
$$B = 0.5\ r_0,$$
$$D = B^2 - A\ C,$$
$$C = -c_f;$$

Other parameters are as follows:

- parameter

$$p = r_0\ (1 + e \cos\phi_0)$$

- are semiaxes ($a > b$)

$$a = p\ (1 - e^2)$$

$$b = \sqrt{a^2 - c^2}$$

- final angle

$$\phi_1 = \phi_0 + 2\alpha_v,$$

where α_v is an angle size of a source being observed from the focal point F (α_v has an algebraic meaning);

$r_1 = p/(1 + e \cos \phi_1)$ is a final radius-vector.

10.2. Hyperbolic zone

This profile is used as additional in the direct ray-tracing scheme, when elliptic zone gives too great intensities. In the proposed algorithm the choice is being carried out among the branches of hyperbola - the near (to the source) and the remote - it depends on required values of illuminance increment.

For the remote branch (being convex at the direction towards a source) the coefficients of quadratic equation [similar to Eqs. (33) and (34)] have the following form (e > 1):

$$A = r_0 \cos\phi_0 + c_f, \quad B = -0.5\, r_0, \quad C = c_f; \tag{35}$$

- parameter

$$p = r_0 (1 + e \cos\phi_0)$$

- are semiaxes (where $c^2 = a^2 + b^2$) (36)

$$a = p (e^2 - 1),$$

$$b = \sqrt{c^2 - a^2}$$

10.3. Parabolic zone

This profile is used to fit the required candlepower curve [1].
We have (e = 1)

$$p = r_0 (1 + \cos\phi_0) \tag{37}$$

$$r_1 = p/(1 + \cos\phi_1)$$

The ray tracing scheme can be crossed and direct (see Table 1 and [1]).

10.4. Constructing profile curve of a zone

Equation of a profile curve in the general coordinate system.

In local coordinate system the equation of a profile curve is as follows:

$$\alpha^2 x_1^2 \pm \beta^2 y_1^2 = 1 \tag{38}$$

where

$$\alpha^2 = 1/a^2 \qquad \beta^2 = 1/b^2$$

(the upper sign refers to ellipse, the lower, to hyperbola).

Equation for parabola

$$y_1 = 0.5(1 - x_1^2 / p) \tag{39}$$

At every i-th step of calculation we need to determine illuminance or candlepower in every subsequent (i+1,i+2,...) interpolating points in order to provide the required value of Ereq.

For this purpose we have to translate Eqs.(38) and (39) in the general coordinate system by:

1) rotation of coordinate system

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

2) translation of origin of coordinates

$$X = x - sx$$

$$Y = y - sy$$

Algebraic equation of the second order curve in Cartesian coordinate system has a classical form:

$$a_{11} x^2 + 2 a_{12} x y + a_{22} y^2 + 2 a_1 x + 2 a_2 y + a = 0 \tag{40}$$

For ellipse and hyperbola the coefficients of Eq.(40) are as follows:

$$a_{11} = 0.5\ ((\alpha^2 \pm \beta^2) + \cos 2v) \tag{41}$$

$$a_{12} = -0.5\ (\alpha^2 \pm \beta^2) \sin 2v$$

$$a_{22} = 0.5\ ((\alpha^2 \pm \beta^2) - (\alpha^2 \mp \beta^2) \cos 2v)$$

$$a_1 = -(a_{11}\, sx + a_{12}\, sy)$$

$$a_2 = -(a_{22}\, sx + a_{12}\, sx)$$

$$a = a_{11}\, sx^2 + a_{22}\, sy^2 + 2\, a_{12}\, sx\, sy - 1$$

where v is an inclination angle between the ellips and the vertical axis OY; (sx,sy) are the focus coordinates in the general system.

For parabola, the translation of the coordinate system gives:

$$a_{11} = \cos^2\alpha \tag{42}$$

$a_{12} = -\sin\alpha \cos\alpha$

$a_{22} = \sin^2\alpha$

$a_1 = p \sin\alpha + sy \sin\alpha \cos\alpha - sx \cos^2\alpha$

$a_2 = p \cos\alpha + sx \sin\alpha \cos\alpha - sy \sin^2\alpha$

$a = (sx \cos\alpha - sy \sin\alpha)^2 - 2 p (sx \sin\alpha + sy \cos\alpha) - p^2$

where α is a current angle, where the zone illuminates.

Polar form.

By substituting $x = r \sin\phi$, $y = r \cos\phi$, we translate Eq.(40) into polar form:

$$F(\phi) = A_{11} r^2 + 2 A_{12} r + A_{22} = 0 \qquad (43)$$

where

$$A_{11} = a_{11} \sin^2\phi + a_{12} \sin 2\phi + a_{22} \cos^2\phi \qquad (44)$$

$A_{12} = a_1 \sin\phi + a_2 \cos\phi$

$A_{22} = a$

Equation (43) sets the function $r = r(\phi)$ and ray-tracing function that is necessary for calculating flashed area of profile curve

$$\alpha(\phi) = \phi + 2\mathrm{arctg}\left(0.5\frac{\delta A_{11}/\delta\phi r + 2\delta A_{12}/\delta\phi}{A_{11} r + A_{12}}\right) \qquad (45)$$

where

$\delta A_{11}/\delta\phi = (a_{11} - a_{22}) \sin 2\phi + 2a_{12} \cos 2\phi$

$\delta A_{12}/\delta\phi = a_{12} \cos\phi - a_2 \sin\phi$

11. Construction of reflector for given illuminance

Inverse problem

The task of constructing profile of specular reflector can be formulated as inverse or identification problem - to find ellipse parameters, an arc P_0P_1 of which produces a prescribed virtual illuminance in a point Q_0 (see Fig.1).

We may suppose that a surface of synthesised reflector is divided on parts by light tubes, and the whole luminous body is seen backward from a given point Q within each tube. Therefore, the number of light tube is equal to the number of points P_S where illuminances E_i have to be provided.

We consider P as an initial point or recurrent point of a profile.
Earlier we analysed in details concentrating properties of ellipse and found out that its amplification of illuminance (with a point source) may grow up to infinity. Infinite values come from δ-like luminance function. For a light source with definite size the luminance amplification can not reach infinity.

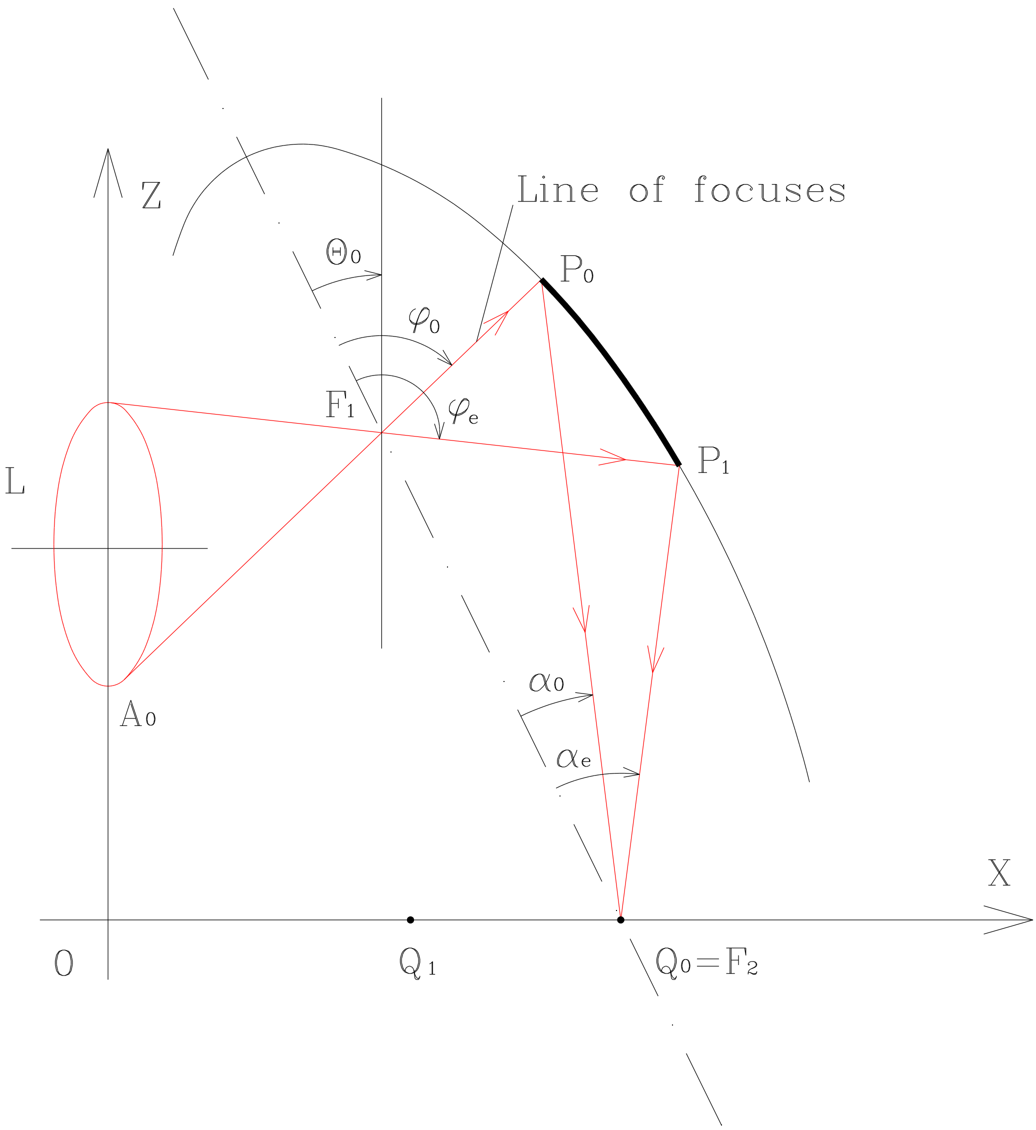

Fig. 1. Constructing of a zone

We further analyse the case of translational symmetry (a flat case). Suppose (see Fig.1) that the first focus F of ellipse slides along steady line $\mathbf{A_0P_0}$ from A_0 *to* P_0. We put it in a vector form

$$\mathbf{F_1} = \lambda\mathbf{A_0} + (\mathbf{1}-\lambda)\mathbf{P_0} \ , \tag{46}$$

where parameter λ runs within $0 \le \lambda \le 1$.
Obviously, a point P_1 moves towards greater angles, and collection angle of luminous flux at a point Q_0 grows as well. This angle (aperture) can be increased thus providing a specified illuminance at Q_0. The line *WP* serves as a physical limit of increasing angle, since point P_1 can not exceed it.
The main steps of algorithm for constructing reflector profile are given below.
A. We find parameters of ellipse that correspond to specified illuminance:
(1) interfocal distance

$$2c = |\mathbf{Q_0F_1}| \ ; \tag{47}$$

(2) initial angle

$$\cos\varphi_0 = \frac{(\mathbf{A_0P_0}\cdot\mathbf{Q_0F_1})}{|\mathbf{A_0P_0}|\cdot|\mathbf{Q_0F_1}|} \ ; \tag{48}$$

(3) final angle of ellipse arc

$$\varphi_f = \varphi_0 + 2\alpha_v \ ,$$

where $2\alpha_v$ is an angular dimension of a luminous body observed from a point F (see Fig. 1).
(4) we find ellipse eccentricity by solving an equation

$$Fe^2 + 2Ee + G = 0 \ , \tag{49}$$

where $F = r_0\cos\varphi_0 + c,\ E = 0.5r_0,\ G = c$; and choose the root from interval 0<e<1.
(5) further parameters of ellipse:
- focal parameter $p = r_0(1 + e\cos\varphi_0)$,
- larger half-axis $a = p(1-e^2)$,
- smaller half-axis $b = \sqrt{pa}$,
- final radius-vector $r_f = p/(1 = e\cos\varphi_f)$,

- angles formed by extreme rays with ellipse axis:

$$\tan(\alpha_0/2) = m\tan(\varphi_0/2)$$
$$\tan(\alpha_f/2) = m\tan(\varphi_f/2) \qquad (50)$$
$$m = (1-e)/(1+e);$$

(31) angle formed by ellipse axis with **OZ-**axis:

$$\cos\theta_0 = \frac{(\mathbf{k}\cdot\mathbf{Q_0F_1})}{|\mathbf{Q_0F_1}|}, \qquad (51)$$

where $\mathbf{k} = [\mathbf{0\,0\,1}]^t$

B. We find virtual illuminance

$$\delta E_0 = L_0(\sin\alpha'_f - \sin\alpha'_0), \qquad (52)$$

where angles of incidence on WP for extreme ray are

$$\alpha'_0 = \alpha_0 - \theta_0,$$
$$\alpha'_f = \alpha_f - \theta_0.$$

We consider equation

$$\delta E_0(\lambda) = \delta E_{giv} \qquad (53)$$

as a non-linear equation with respect to unknown λ. It can be solved with the aid of numerical methods, and Eqs.(21-27) constitute an iteration cycle. Solution is found with ε-accuracy, thus, it identifies "the best" ellipse from one-parameter family of ellipses, which foci are located on steady line A_0P_0.

Direct problem

When current part of profile is determined, it is followed by the next segment, and so on. Therefore, each segment or zone adds illuminance in test points. So, on each step of profile construction we need to correct specified values of illuminance δE_{giv}.

Here follows the direct problem: under given vector of parameters of ellipse segment, found for the point Q_f, and given light source - to calculate illuminance at the points $Q_i, i = k+1, k+2, \ldots$ Luminous body is described by expression:

$$FL(\mathbf{s}, \mathbf{b}) = \mathbf{0} \ , \tag{54}$$

where $\mathbf{s}$ is a point vector, $\mathbf{b}$ is a vector of parameters.
It is convenient to compute illuminance in general coordinate system. Here, ellipse parameters have to be transformed, hence vector of ellipse parameters takes a form

$$\mathbf{R} = \left[a_{11}\ a_{12}\ a_{22}\ a_{13}\ a_{23}\ a_{33}\right]^t .$$

Besides Monte Carlo methods, inverse-ray-tracing method is most efficient for direct calculations of illuminance. We give a brief description of this method as applied to our problem:

(1) specify the direction of inverse ray

$$\mathbf{a} = \left[\sin\xi \quad 0 \quad \cos\xi\right]^t ;$$

(2) form an inverse ray from point Q_f

$$\mathbf{s} = \mathbf{Q_f} + \mathbf{a} \cdot l \ , \tag{55}$$

(3) knowing $\mathbf{R}$ and RTF for a zone, find a normal at a point of intersection $\mathbf{P}$ of an inverse ray and a mirror, and calculate a matrix of reflection $\mathbf{M_r}$,

(4) find direction of reflected (back-passing) ray

$$\mathbf{a}' = \mathbf{M_r} \cdot \mathbf{a} \ , \tag{56}$$

(5) form a reflected ray from a point $\mathbf{P}$

$$\mathbf{s}' = \mathbf{P} + \mathbf{a} \cdot l \ , \tag{57}$$

where l is unknown parameter,

(6) find a meeting point of reflected ray $\mathbf{s}'$ with a surface of luminous body

$$FL(\mathbf{s}, \mathbf{b}) = \mathbf{0} \ . \tag{58}$$

Equation (48) can be solved numerically with respect to *l.* If a luminous body is described by 2nd-order equation, Eq.(48) is quadratic having a discriminant D

if $D \geq 0$ - point $\mathbf{P}$ is considered bright;

if $D < 0$ - point $\mathbf{P}$ is considered dark.

In this way boundaries of flashed area for selected point Q_i on a working plane are calculated, and an aperture of a ray beam coming to selected point from reflector zone.
Note, that found at each step point $\mathbf{P_1}$ serves as internal point of a flashed area for the next point on a working plane. Therefore, direction of reflected ray does not change here, and the zones join smoothly.

12. Equation for calculating zone flashed area

Let $\alpha(\phi)$ be the ray-tracing function, then the conditions that arbitrary point (r,ϕ) sends the light toward the point x_i are as follows:

$$F_1 = \alpha(\phi) - \mathrm{arctg}((r \sin\phi - x_i)/(r \cos\phi + H)) + \alpha_{ac} \geq 0$$

$$F_2 = \alpha(\phi) - \mathrm{arctg}((r \sin\phi - x_i)/(r \cos\phi + H)) - \alpha_{cl} \leq 0 \qquad (59)$$

where a_{ac} is the angular dimension of source luminous body observed from the given point and estimated counter-clockwise from the axial (central) line; α_{cl} is the angular dimension counted clockwise from the axis. For cylinder $\alpha_{ac} = \alpha_{cl} = \mathrm{asin}(R/r(\phi))$ Equation $F_1(\phi)=0$ and $F_2(\phi)=0$ determine the boundary points of zone flashed area.

In order to calculate the candlepower at the direction α_i, we have

$$F_1 = \alpha(\phi) - \alpha_i + \alpha_{ac} \geq 0$$

$$F_2 = \alpha(\phi) - \alpha_i - \alpha_{cl} \leq 0 \qquad (60)$$

13. Step of searching flashed area boundaries

Since direct calculation is based on search method (i.e. inverse-ray method), selection of step in determining flashed area boundaries counts a lot for the time of computation. Flashed area boundaries (image) in the main section of reflector $\psi=0$ are described by equation

$$\alpha(\varphi) - \alpha_v \pm a_\varphi = 0 \ , \qquad (61)$$

where $\alpha(\varphi)$ is a ray-tracing function (RTF); α_v is an observation angle; a_φ is an angular dimension of luminous body.

We assume for simplicity that luminous body is a globe with radius R, then

$$a_\varphi = \arcsin(R / r),$$

where $r = r(\varphi)$ is a radius-vector of reflector profile curve.

From Eq.(61) we obtain an evaluation of minimal interval

$$h_\varphi = \min \frac{2a}{\left|\dfrac{d\alpha}{d\varphi}\right|} \ . \qquad (62)$$

Accounting that image moves towards edge, we get

$$h_\varphi = \frac{\min(a_0, a_1)}{\max\left|\dfrac{d\alpha}{d\varphi}\right|} \ , \qquad (63)$$

where $a_0 = a(\varphi_s), \quad a_1 = a(\varphi_e)$.

We designate $K_\varphi = \left|\dfrac{d\alpha}{d\varphi}\right|$, then accounting parabolic points of reflector ($K_\varphi =$), get

$$h_\varphi = \min(1, K_\varphi) * \min(a_0, a_1) \ . \qquad (64)$$

14. Selection of envelope shape

In case when the calculation is carried out for multiple sources, the concept of envelope is introduced.

Let $x_{max}, x_{min}, y_{max}, y_{min}$ be the boundary parameters of a rectangular P enveloping the sources being set.

If

$$0.5 < |x_{max}-x_{min}|/|y_{max}-y_{min}| < 2 \qquad (65)$$

then the envelope is considered to be a circle with a diameter being equal to the diagonal of P. In other case the envelope is considered to be a rectangular P. When synthesizing a surface, the assembling of a prescribed curve is being carried with respect to the envelope.

The zonal candlepower curve is calculated by applying the method of inverse ray-tracing [1] and accounting the real geometry. At this stage the obtained ray-tracing function $\alpha(\varphi)$ can be used.

15. Functions of flux distribution

When given light distribution is presented in relative values an approximate flux balance should be calculated. Functions considered below are used for definition of flux from the light source.

Cylinder:

$$F(\phi) = I_0\, \phi, \qquad 0 \le \phi \le 2\pi$$

where

$$I_0 = 2\, R\, l\, L_0$$

Rectangle:

$F(\phi) =$

$$I_{01}(1 - \cos\phi) + I_{02} \sin\phi, \qquad 0 \le \phi \le \pi/2$$

$$I_{01}(1 - \cos\phi) + I_{02} (2 - \sin\phi), \qquad \pi/2 < \phi \le \pi$$

$$I_{01}(3 + \cos\phi) + I_{02} (2 + |\sin\phi|), \qquad \pi < \phi \le 3\pi/2$$

$I_{01}(3 + |\cos\phi|) + I_{02}\,(4 + \sin\phi)$, $\quad 3\pi/2 < \phi \leq 2\pi$

where

$I_{01} = 2\ b\ l\ L_0 \qquad I_{02} = 2\ a\ l\ L_0$

Stripe:

$F(\phi) =$

$I_0 \sin\phi$, $\quad 0 \leq \phi \leq \pi/2$

I_0, $\quad \pi/2 < \phi \leq 3\pi/2$

$I_0\,(1 + \sin\phi)$, $\quad 3\pi/2 < \phi \leq 2\pi$

where

$I_0 = 2\ a\ l\ L_0$

REFERENCES

1. Kusch O. Computer-Aided Optical Design of Illumination and Irradiating Devices, ASLAN, Moscow, 1993.