SHAPE 4

Reflector Design Software

Mathematical Description

Point Light Source

Reflector Design Research, Inc
www.reflector-design.com
info@reflector-design.com

Contents

INTEGRATED CONCEPT FOR COMPUTATION AND DESIGN OF SPECULAR REFLECTORS

1. Introduction

Integrated concept for computation and design of specular reflectors for lighting devices is based on terms of geometrical optics and an idea of a point source. This includes in itself both direct and inverse problems. Final equations for direct problems can be found here that express intensity through differential properties of a reflecting surface, as well as through Gaussian curvature of the latter and source radiation indicatrix. Inverse problems are being solved by the way of inverting these equations (except for the singular points). It makes designing of reflectors being a closed system. The Cauchy problem for the second-order differential equation thus has to be solved, and that provides an accuracy and uniqueness of solution.
The general concept of the geometrical-and-optical approach permits to investigate and synthesise various mirror systems in the most simple way. One of the useful properties of a software package constructed on this principle is the mutual control of calculations carried out on direct and inverse problems.

2. Mathematical description of a problem

2.1. Equation for calculating intensity of reflected light

We consider the solution for direct problems, i.e. determination of illuminance (luminous intensity) created by reflector at a node *(X,Y)* of a reference plane. As is known from [1], this problem for a point source is reduced to calculating the Jacobian D of the transformation of curvilinear coordinates *u, v* chosen on reflector surface to the coordinates on a reference plane:

$$D = \begin{vmatrix} X_u & Y_u \\ X_v & Y_v \end{vmatrix} .$$

The value of D characterises the section area of a reflected ray beam in its normal section, or the beam divergence. According to the Malus' theorem [2] such section always exists for a point source.
The choice of coordinate lines $u = const$, $v = const$ is mostly determined by the kind of an optical system and the type of reflecting surface. We consider either axially symmetric system or a surface with translational symmetry (cylindrical system) only. Here it is convenient to enter spherical coordinates: (where $u = \varphi$ is a polar angle, $v = \psi$ is an azimuthal angle), and $r = r(x, y, z)$ is a radius vector of a surface point (Fig.1).

We make the following definitions [1]:
- a point **P** : $[xyz]^t = [r\sin\varphi\cos\psi\, r\sin\varphi\sin\psi\, r\cos\varphi]^t$;

- an incident ray: $\hat{l} = [\sin\varphi\cos\psi\sin\varphi\sin\psi\cos\varphi]^t$;
- a normal to a surface: $\hat{n} = [n_x n_y n_z]^t = [\sin\delta\cos\beta\sin\delta\sin\beta\cos\delta]^t$;

[1] "t" means transposition

- a reflected ray: $\hat{l}' = \begin{bmatrix} \sin\varphi\cos\psi - 2n_x\cos\theta \\ \sin\varphi\sin\psi - 2n_y\cos\theta \\ \cos\varphi - 2n_z\cos\theta \end{bmatrix}$

where $\cos\theta = (\hat{n}, \hat{l}) = \cos\delta\cos\varphi + \sin\delta\sin\varphi\cos\psi, \theta$ is an angle of ray incidence on a mirror element.

Let the task plane (TP) be parallel to the plane XOY and located at a distance of $Z=h$ from the latter.

The trace of reflected ray on the TP is defined by vector

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} x + \sigma*(\sin\varphi\cos\psi - 2n_x\cos\theta) \\ y + \sigma*(\sin\varphi\sin\psi - 2n_y\cos\theta) \end{bmatrix}, \qquad (1)$$

where $\sigma = (r\cos\varphi - h)/(\cos\varphi - 2n_z\cos\theta)$ (2)

is the distance between a reflector point and a point $[X, Y]^t$.

Now we apply the conditions of symmetry to equations (1,2).

2.1.1. Rotational symmetry

While determining the coordinates of a reflected beam on a task plane we note that in this case the normal lies in a meridian section of a surface, i.e. $\beta = \psi$, and the expression for σ acquires the following kind

$\sigma = (r\cos\varphi - h)/\cos\alpha$,

where $\alpha = 2\delta - \varphi$ is a polar angle of reflected ray [1].

We designate $X_0 = X_{\psi=0}$, in this case

$$\begin{bmatrix} \mathrm{X} \\ \mathrm{Y} \end{bmatrix} = \mathrm{X}_0 \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix}, \qquad (3)$$

where

$$X_0 = r\sin\varphi - (r\cos\varphi - h)\tan\alpha \ . \qquad (4)$$

Since **D** has physical meaning being invariant during rotation of coordinate system, hence

$$D = D_{\psi=0} = \frac{dx}{d\varphi} X_0 \ . \qquad (5)$$

Taking into account the equations

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[1] We take the following rule of signs: angles counted clockwise are considered to be positive, and if anticlockwise - negative.

$$\frac{d}{d\varphi}(r\sin\varphi) = r\frac{\cos\delta}{\cos\theta} \quad ,$$

$$\frac{d}{d\varphi}(r\cos\varphi) = -r\frac{\sin\delta}{\cos\theta} \quad ,$$

where $\theta = \dfrac{\varphi-\alpha}{2}$, we obtain

$$\frac{dX}{d\varphi} = \frac{r}{\cos\alpha}(1-\frac{\sigma}{r}\frac{d\alpha}{d\varphi}) \ . \qquad (6)$$

Substituting (6) into (5) we obtain the expression for the determinant[1]

$$D = \frac{r^2\sin\varphi}{\cos\alpha} * D_1 * D_2 \ , \qquad (7)$$

where

$$D_1 = 1-\frac{\sigma}{r}\frac{d\alpha}{d\varphi},$$

$$D_2 = 1-\frac{\sigma}{r}\frac{\sin\varphi}{\sin\alpha}$$

Computation of illuminance.

Expression for computing illuminance [1] can be designated in the following form (we suppose factor of reflection to be equal to 1, for convenience):

$$E_h = J(\varphi)\frac{\sin\varphi}{D} \quad , \qquad (8)$$

where $J(\varphi)$ is a luminous intensity of a source at the direction of φ.
Still, it is more easy to operate with normal illuminance if we suppose that $E_n = E_h / \cos\alpha$. Accounting (6), (7), and (8) we obtain

$$E_n = E_0 * k_1 * k_2 \ , \qquad (9)$$

where $E_0 = J(\varphi)/r$ is an illuminance in the normal section of a light tube near a mirror. Obviously, if we operate with a point source, this section or the wavefront appears to be a spherical surface having a centre at the position of a light source.

$$k_1 = D_1^{-1} = (1-\frac{\sigma}{r}\frac{d\alpha}{d\varphi})^{-1},$$

$$k_2 = D_2^{-1} = (1-\frac{\sigma}{r}\frac{\sin\varphi}{\sin\alpha})^{-1}$$

[1] It should be noted that in expression for D it is necessary to choose sign (+) or (-) depending on the scheme of ray course (elliptic or hyperbolic type [1]).

The convenience of Eq.(9) is in direct expression of illuminance through ray-tracing function (RTF) - $\alpha(\varphi)$, and its first derivative (DTF) - $d\alpha / d\varphi$. Ray-tracing function is widely applied in computations of lighting devices with point sources, as well as with sources with finite dimensions [1,3]. The values k_1, k_2 are the principal curvatures in the normal section of a beam both in meridian and sagittal planes, thus $k = k_1 k_2$ is the total or Gaussian curvature in the normal section of a ray beam falling on the TP [1,2].

If the value of E_0 on any reference wavefront surface is known, then illuminance at the distance σ from it will be equal to [2]:

$$E_n = E_0 * k_1 * k_2, \tag{10}$$

where

$$k_1 = 1 + \frac{\sigma}{R_1'}, \quad k_2 = 1 + \frac{\sigma}{R_2'} \quad .$$

R_1' and R_2' are the main radii of curvature of reference surface (Fig.2).
Taking into account Eq.(9), we find these radii

$$R_1' = -r\left(\frac{d\alpha}{d\varphi}\right)^{-1} \quad , \quad R_2' = -r\left(\frac{\sin\varphi}{\sin\alpha}\right)^{-1} . \tag{11}$$

Computing of luminous intensity

Luminous intensity is determined on infinitely large distance from where an illuminating surface can be treated as a point source. In this case a beam wavefront of straight-line rays is a sphere, hence, according to the law of reciprocal square distances [1,2]

$$I_\alpha \to E_n * \sigma^2 \ \mathit{under} \quad \sigma \to \infty.$$

The total curvature from Eqs.(9,10) will be

$$k = k_1 * k_2 = \left(\frac{d\alpha}{d\varphi}\right)^{-1} * \left(\frac{\sin\varphi}{\sin\alpha}\right)^{-1} . \tag{12}$$

A luminous intensity of a mirror we shall rate to a luminous intensity of a source, assuming an amplification factor equals

$$M_I = \frac{I(\alpha)}{I(\varphi)} = \left(\frac{d\alpha}{d\varphi}\right)^{-1} * \left(\frac{\sin\alpha}{\sin\varphi}\right)^{-1} . \tag{13}$$

It is convenient to present an expression that links illuminance with luminous intensity

$$E_n = I(\alpha) * (\sigma + R_1')^{-1} * (\sigma + R_2')^{-1} \tag{14}$$

where R_1' and R_2' are the principal curvature radii being found from Eq.(11).
Expression (14) shows that the wave reflected from a mirror is generally not spherical, thus both curvature radii must be taken into account. In this case, obviously, the following expressions will serve as a criteria of validity for the law of reciprocal squares

$\sigma >> R_1'$ and $\sigma >> R_2'$.

2.1.2. Translational symmetry

Normal vector entirely lies at the plane XOY, i.e. $\beta = 0$.
We obtain

$\hat{n} = [\sin\delta \quad 0 \quad \cos\delta]^t$;

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (r\sin\varphi - \sigma\sin\alpha)\cos\psi \\ (r+\sigma)\sin\psi \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} * \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} \tag{15}$$

where $\sigma = (r\cos\varphi - h)/(\cos 2\delta\cos\varphi + \sin 2\delta\sin\varphi\cos\psi)$.

A planar wave

We put the problem as planar if we assume that the flux spreads at a plane parallel to the plane ZOX. This corresponds to a planar wave of a linear source extended along a focal axis of reflector. We select a principal section assuming $\psi = 0$ in Eq.(12).
Illuminance. Similar to spatial problem we obtain

$$E_n = E_0 * k_1, \tag{16}$$

where

$$E_0 = J(\varphi)/r \quad ,$$

$$k = D^{-1} = \left(\frac{dx}{d\varphi}\right)^{-1} \tag{17}$$

or

$$k = [1 - (\sigma/r)*(d\alpha/d\varphi)]^{-1} . \tag{18}$$

Expressing illuminance by curvature radius of reference front we obtain a slightly different kind of equation for E

$$E_n = E_0 * k_1, \tag{19}$$

where

$$k_1 = (1 + \sigma/R_1')^{-1},$$

$$R_1' = -r*(d\alpha/d\varphi)^{-1} .$$

Luminous intensity. Taking into account Eq.(16) through passage to the limit we obtain an amplification factor

$$M_I = I(\alpha)/I(\varphi) = (d\alpha/d\varphi)^{-1} . \qquad (20)$$

Similar to Eq.(14) a relation between luminous intensity and illuminance takes a form

$$E_n = I(\alpha)*(\sigma + R_1')^{-1} . \qquad (21)$$

Thus, an illuminance, at least for distances $\sigma >> R_1'$, changes in accordance with the law of first-order reciprocals of distance in contrast to the case with rotationally symmetric reflector. In other principal plane $I(\alpha)$ will follow a cosine law.
Such kind of reflectors is applied if a diagram with very different petal widths in principal sections has to be created.

A spherical wave

Assuming a source to be a point, we find a flux distribution in the main section $\psi = 0$. In this case (see Eq.(12))

$$D = D_{\psi=0} = \frac{dA}{d\varphi} * B = \frac{dx}{d\varphi}(r + \sigma) . \qquad (22)$$

Illuminance equals

$$E_n = J(\varphi) * \left(\frac{dx}{d\varphi}\right)^{-1} * (r + \sigma)^{-1} \qquad (23)$$

or

$$E_n = E_0 * k_1 * k_2 , \qquad (24)$$

where $E_0 = J(\varphi)/r^2$ is an illuminance of reference sphere near a mirror,
$k_1 = [1 - (\sigma/r)(d\alpha/d\varphi)]^{-1}$ and
$k_2 = [1 + (\sigma/r)]^{-1}$ are curvatures in normal section of reflected beam.

Luminous intensity

Quite similar to the case of planar wave (Eqs.(20,21)), we have

$$M_I = I(\alpha)/J(\varphi) = (d\alpha/d\varphi)^{-1} , \qquad (25)$$
$$E_n = I(\alpha)*(\sigma + R_1')^{-1}*(\sigma + R_2')^{-1} , \qquad (26)$$

where[1] $R_1' = -r(d\alpha/d\varphi)^{-1}$, $R_2' = r$.

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[1] Note that $R_1' = 1/k_1!$.

We observe that the curvature radius of reference wavefront in meridian section extends in $(d\alpha / d\varphi)^{-1}$ times, while in other section, where curvature line is straight, curvature radius does not increase.

2.1.3. Ray-tracing function

Calculation of illuminance and luminous intensity is based on information about ray-tracing function RTF and its derivative DTF that were found for given generatrix profile $r = r(\varphi)$ of a surface. These functions can be found from the equation of symmetrical specular surface [1]:

$$\frac{dr}{rd\varphi} = \tan\,\theta\,, \qquad (27)$$

where $\theta = 0.5(\varphi - \alpha)$.
Converting and differentiating Eq.(21), we obtain the general equations for RTF and DTF

$$\alpha = \varphi - 2arctg\left(\frac{dr}{rd\varphi}\right)\,, \qquad (28)$$

$$\frac{d\alpha}{d\varphi} = 1 - 2\left[\frac{d^2 r}{d^2\varphi} * \cos^2\theta - \sin^2\theta\right]. \qquad (29)$$

Example: Let us take a reflecting sphere with radius R that contains a source in its centre. Equation for sphere is r=R, thus in accordance with Eqs.(28,29) we obtain $\alpha = \varphi \; \frac{d\alpha}{d\varphi} = 1.$
Following described method we find that a sphere illuminates like a point source:

$$M_I \equiv 1,$$
$$E_n = E_0 * (1 - \sigma / R)^{-2}, \quad (E_0 = J(\varphi) / R^2).$$

Equations (28) and (29) can be used to finding RTF when the profile is set in table form. In this case the profile is interpolated by a suitable analytical function which later is being differentiated [1,4].

The form of RTF is determined by optical properties of a surface and by position of a point source which is usually coincided with a focal point of an optical surface. It is easy to see that assigning of RTF and DTF will wholly determine RTF and DTF for any displacement of a source from focus.
For any point $(\varphi, r(\varphi))$ the following may be regarded as being known:

- a normal $\delta = 0.5 * (\varphi + \alpha)$;
- a curvature $k = 0.5 * \left[1 + (d\alpha / d\varphi)\right] / r.$

If a new position of a source is prescribed by a displacement vector $\left[d_x \; 0 \; d_z\right]^t$, then its novel polar coordinates in the plane XOZ are as follows

$$r' = \sqrt{\left[(x-d_x)^2 + y^2 + (z-d_z)^2\right]},$$
$$\varphi' = \arctan\left[(x-d_x)/(z-d_z)\right]. \tag{30}$$

Algorithm of recalculating of RTF and DTF for a novel position of a source $(\varphi', r'(\varphi'))$ is as follows

(1) $\alpha' = 2\delta - \varphi';$
(2) $\theta' = 0.5*(\varphi' - \alpha');$
(3) $(d\alpha/d\varphi)' = 2kr'/\cos\theta' - 1.$ (31)

2.1.4. Singular points in equations for calculating intensity

It is a known fact that the concepts of geometrical optics are adequate if a wave amplitude at a given point varies not too fast [2]. Presence of caustic points makes us to proceed to averaging in calculated field [4]. Their presence can be found out by assuming *D=0* in equations for intensity.

Rotational symmetry

Assuming in Eq.(7) *D=0* we obtain:

$$\sigma = r\left(\frac{d\alpha}{d\varphi}\right)^{-1},$$
$$\sigma = r\left(\frac{\sin\alpha}{\sin\varphi}\right)^{-1}, \tag{32}$$

that is, we have focal points in illuminated field [1].
It is easy to find in Eq(32) also the expressions for curvature radii of a reference surface, but taken with opposite sign.
A point $\varphi = 0$ is also a singular point for axially symmetrical system.
If $\varphi = 0,\ \ \alpha = 0$, *and* $d\alpha/d\varphi \neq 0$, that is, we have an umbilical point in the naught, then the value of k in the equations for illuminance (7-14) have to be redetermined as follows:

$$k_2 = k_1 = \left[1 - \left(\frac{\sigma}{r}\right)\left(\frac{d\alpha}{d\varphi}\right)\right]^{-1}, \tag{33}$$

and in Eq.(13) for luminous intensity

$$k_2 = k_1 = \left(\frac{d\alpha}{d\varphi}\right)^{-1}. \tag{34}$$

If for $\varphi = 0$ we obtain $\alpha = 0$ (a point of return) then this point must be excluded from the consideration or be redefined by averaging illuminance over regarded region. In case of illuminance, parabolic points, i.e. points where $d\alpha/d\varphi = 0$, act as analogue for focal points.

Translational symmetry

Singular points are elliptical (for illuminance) and parabolic (for luminous intensity). Umbilical points do not exist. Since for plane $k = 1 + (\sigma / r)$, singular points do not appear in the case of such reflector (faceted).

2.1.5. Examples of computation

(1) Ellipsoid (with a source positioned in focal point (Fig.3))

Ray tracing

We have the following expressions for RTF and DTF [1]:

$$\tan\left(\frac{\alpha}{2}\right) = \frac{1-e}{1+e}\tan\left(\frac{\varphi}{2}\right), \tag{35}$$

$$\frac{d\alpha}{d\varphi} = \frac{\sin\alpha}{\sin\varphi} = \frac{1-e^2}{1 \pm 2e\cos\varphi + e^2}, \tag{36}$$

where e is an ellipse eccentricity. Obviously, $e<1$ for ellipse, and reflected rays converge toward optical axis ($\alpha>0$); $e>1$ for hyperbola, and reflected rays diverge from optical axis ($\alpha<0$); $e=1$ in case of parabola.

Intensity

An amplification factor for ellipsoid (or hyperboloid) is as follows

$$M_I = \frac{I(\alpha)}{J(\varphi)} = \left(\frac{1 \pm 2e\cos\varphi + e^2}{1-e^2}\right)^2 . \tag{37}$$

Minus should be taken in front of the second member of numerator for convex hyperbola.
To find illuminance at a distance σ it is convenient to use the form of Eq.(14)

$$E_n = I(\alpha) * (\sigma + R')^{-2}, \tag{38}$$

where $R' = -r\left(\frac{\sin\alpha}{\sin\varphi}\right)^{-1} = -r\frac{(1 \pm 2e\cos\varphi + e^2)}{(1-e^2)}$.

We denote

$$m = (1 \pm 2e\cos\varphi + e^2)/(1-e^2).$$

If $\sigma = -R = r*m$, then we have a conjugate focus, that is $r^* = r*m$ and luminous intensity amplification factor is

$$M_I = m^2 = (r^* / r)^2. \quad (39)$$

Expression (39) becomes rather evident if we account the following:
(1) values of illuminance on reference surfaces s^* and s (Fig.3) are equal;
(2) a source being placed at a conjugate focus (r^*) should make the same illuminance on ellipse element as a source being placed at the first focus

Variation range of luminous intensity

(a) Ellipsoid (Fig.3)

Evidently, accounting Eq.(39), and for angle range $0 \le \varphi \le \pi$ the value of $M(\varphi)$ is a descending periodic function that varies from $M_{\max} = \left[(e+1)/(e-1)\right]^2$ to $M_{\min} = 1/M_{\max}$; $M_I = 1$ for $\varphi^* = \varphi = \pi - \arccos(e)$.

(b) Hyperboloid (Fig.4)

Concave branch (Fig.4a)
In the angle range $0 \le \varphi \le \varphi^*$ the function $M(\varphi)$ is descending and varies from $M_{\max}$ to 1, while $\varphi^* = \pi - \arccos(e)$.

Convex branch (Fig.4b)
In the angle range $0 \le \varphi \le \varphi^{**}$ the function $M(\varphi)$ is ascending and varies from $M_{\min}$ to 1, while $\varphi^{**} = \arccos(1/e)$. The picture opposite to the case of concave hyperboloid is observed: a ray is less intensive for the centre and has maximal intensity to the edge of reflector. This property is applied in design of Cassegrain systems [1].

Variation range of illuminance

If illuminance values are rated to illuminance of reference surface $E_0 = I(\alpha)/R'^2$, then illuminance amplification produced by ellipsoid is

$$M_E = E_n / E_0 = (1 + \sigma / R')^{-2}, \quad (40)$$

where R' is specified from Eq.(38).

In accordance with Eq.(40) illuminance varies from 0 to ∞ for ellipsoid, and from 0 to 1 for hyperboloid. That is M_E describes concentrating properties of the 2nd-order curves.

(2) Paraboloid (with a point source positioned in focus (Fig.2))

Ray tracing

For *e=1* Eqs.(35,36) give the following: $\alpha(\varphi) \equiv 0, \quad d\alpha / d\varphi \equiv 0$.

Intensity

According to Eqs.(37,39) amplification of paraboloid gives $M_I = \infty$. Formally it can be explained by infinite luminance of a point source having a null area. From Eqs.(7) and (40) we obtain that divergence of a beam does not change, i.e. $D \equiv 1$, $M_E \equiv 1$.
Consequently, after reflection from paraboloid surface

$$E_n = E_0 = J(\varphi)/r^2,$$

or

$$E_n = (J(\varphi)/f^2)*(1+(X/2f)^2)^{-2} , \qquad (41)$$

where *f=0.5p* is a focal distance of parabola. It follows from Eq.(41) that paraboloid creates four times greater illuminance than a flat plane placed at a distance *h=f* from a point source.
Since both curvature radii in Eq.(11) are infinite, illuminance does not depend on the distance to the reference plane. Hence, parabola turns spherical wave passing from a source into a plane wave.
Parabola does not form caustics, thus a rather uniform illuminance is provided at the field near optical axis.

We have the following expressions in case of parabola:

- radius-vector $r = 0.5p\cos^{-2}(0.5\varphi)$;
- curvature in meridian section $\chi_1 = p^{-1}\cos^3(0.5\varphi)$;
- curvature in sagittal section $\chi_2 = p^{-1}\cos\ (0.5\varphi)$;
- Gaussian curvature $\chi = 4r^2$;
- ray *X*-coordinate on a TP $x = p\tan(0.5\varphi)$;
- the first derivative from X $\dfrac{dX}{d\varphi} = r$.

These formulas may be useful when testing the programs of computing illuminance and luminous intensity produced by specular surfaces with a source displaced from system focus (see above or [1]).

(3) Plane mirror with filament source

Ray tracing

Polar equation of a straight line is as follows:

$$r = p/\cos(\varphi - \delta_0) ,$$

where p is a parameter (the least distance from a line to a centre), δ_0 is an angle of normal to a mirror plane.
RTF and DTF for a plane mirror will be as follows:

$$\alpha = \delta_0 - \varphi$$
$$\frac{d\alpha}{d\varphi} = -1 \qquad . \qquad (42)$$

Intensity

Evidently, luminous intensity can not exceed a luminous intensity of a light source, i.e. $M_I = 1$. We obtain for illuminance (since $E_0 = I(\varphi)/r$)

$$M_E = \left[1 + (\sigma / r)\right]^{-2}. \tag{43}$$

As it follows from Eq.(43) caustics in reflected light are absent.

(4) Parabolic cylinder with a point source

In accordance with parabola property

$$r + \sigma = 2f + h = C,$$

where C is a constant, p is parabola parameter, h is a distance between focus and TP. From Eqs.(23,24) we obtain

$$\begin{aligned} &M_I = \infty, \\ &M_E = E_n / E_o = C, \end{aligned} \tag{44}$$

where $E_0 = J(\varphi)/r$, i.e. we obtain distribution curve of illuminance similar to that for a plane (similarity factor is equal to C).

Tables 1 and 2 show some lighting characteristics that give evidence about concentrating properties of optical mirror surfaces.

Table 1

Kind of surface	*Characteristics*				
	Angle range	*RTF*	*DTF*	M_I	M_E
Ellipsoid	$[0, \pi)$	$2\arctan(\frac{1}{m}\tan\frac{\varphi}{2})$	$\sin\alpha / \sin\varphi$	$m^2 \div m^{-2}$	∞
Hyperboloid - *concave branch* - *convex branch*	$[0, \varphi^*)$ $[0, \varphi^{**})$	$2\arctan(\frac{1}{m}\tan\frac{\varphi}{2})$ $2\arctan(\frac{1}{m}\tan\frac{\varphi}{2})$	$\sin\alpha / \sin\varphi$ $\sin\alpha / \sin\varphi$	$1 \div m^{-2}$ $m^{-2} \div 1$	$0 \div 1$ $0 \div 1$
Paraboloid	$[0, \pi)$	0	0	∞	1
Plane mirror	$(\delta_0 - \pi/2, \delta_0 + \pi/2)$	$-\varphi$	-1	1	$0 \div 1$
Sphere	$[0, \pi]$	φ	1	1	$0 \div 1$

The following designations are used in Tables 1 and 2:

m=(e+1)/(e-1);
$\varphi^{*} = \pi - \arccos(1/e)$;
$\varphi^{**} = \arccos(1/e)$;
δ_0 is a polar angle to a straight line;
ρ is a reflection factor ;
$\tan\,\alpha = (r\sin\,\varphi - X)/(r\cos\,\varphi + h)$ is a polar angle of reflected ray;
$\sigma = (r\cos\,\varphi + h)/\tan\,\alpha$ is a distance between a mirror point and TP;
a double sign specifies selection of RTF scheme.

Concentrating and dispersing properties of conic sections are applied in design (synthesis) of reflectors with sources of finite dimensions [1].

2.2. Inverse problem

2.2.1. Equations for profile curve of reflector

Equations (5)-(8) and (16)-(17), as well as Eq.(29) show that operator acting at a space of $r(\varphi)$-functions is the 2nd-order differential operator. In order to make it invertible, we have to exclude singular points, i.e. the points where *D=0.*
Having defined initial conditions: $\varphi = \varphi_0$ leads to $r = r_0, \alpha = \alpha_0,$ *or* $x = x_s$, - we come to Cauchy problem for a differential equation of the second order or to the equivalent system of two equations of the first order. In essence we rewrite expressions for luminous intensity $I(\alpha)$ and illuminance *E(x)* that were obtained earlier up-side-down by leaving differentials at the left part, while putting demanded functions $I(\alpha)$ and *E(x)* to the right. Thus, we obtain the equations for calculating a profile curve of specular reflector (Eq.(45) in Table 2).

Table 2

Kind of source and desired characteristic	*Reflector with rotational symmetry (Eq.(45))*	*Reflector with translational symmetry (Eq.(45))*
Point source *Illuminance*	$\frac{d\varphi}{dX} = \pm\frac{X * E(X)}{\rho * J(\varphi) * \sin\varphi}$ $\frac{dr}{dX} = \frac{d\varphi}{dX} r\tan(\frac{\varphi-\alpha}{2})$	$\frac{d\varphi}{dX} = \pm\frac{E(X) * (r+\sigma)}{\rho * J(\varphi)}$ $\frac{dr}{dX} = \frac{d\varphi}{dX} r\tan(\frac{\varphi-\alpha}{2})$
Point source *Luminous intensity*	$\frac{d\alpha}{d\varphi} = \pm\frac{\rho * J(\varphi) * \sin\varphi}{I(\alpha)\sin\alpha}$ $\frac{dr}{d\varphi} = r\tan(\frac{\varphi-\alpha}{2})$	$\frac{d\alpha}{d\varphi} = \pm\frac{\rho * J(\varphi)}{I(\alpha)}$ $\frac{dr}{d\varphi} = r\tan(\frac{\varphi-\alpha}{2})$
Linear source *Illuminance*		$\frac{d\varphi}{dX} = \pm\frac{E(X)}{\rho * J(\varphi)}$ $\frac{dr}{dX} = \frac{d\varphi}{dX} r\tan(\frac{\varphi-\alpha}{2})$
Linear source *Luminous intensity*		$\frac{d\alpha}{d\varphi} = \pm\frac{\rho * J(\varphi)}{I(\alpha)}$ $\frac{dr}{d\varphi} = r\tan(\frac{\varphi-\alpha}{2})$

The right parts of Eqs.(45) are the functions in two variables: X-coordinate of a ray on a TP and a polar angle φ, or the coordinates α and φ. Solution of these equations exists and it is unique within sufficiently broad limits [6].

2.2.2. Accounting of direct light of a source

If we take into account the direct light of a source, then the uniqueness of solution may vanish. We take advantage of the fact that one of the equations in Table 2 is the equation with separable variables. Integrating this equation we obtain, so to say, the equation of flux balance:

$$\pm\rho\int_{\varphi_s}^{\varphi}\sin\varphi * J(\varphi)d\varphi = \int_{X_s}^{X}\left[E(X) - E_d(X)\right]XdX, \qquad (46)$$

where *E(X)* is a prescribed curve of illuminance distribution; $E_d(X)$ is a direct illuminance at the field of definition of illuminance curve $[X_s, X_e]$ (see Fig.). This equation uniquely defines ray-tracing function in the whole range except for those points where the right-side expression under the integral turns to null. Therefore, we obtain an equation for critical points:

$$M * e(X) = E_d(X),$$

where *e(X)* is a prescribed illuminance curve (in relative units), *M* is an amplification factor. Multiplier *M* is found from expression

$$M = \frac{\int_{\varphi_0}^{\varphi_1} J(\varphi)\sin\varphi\, d\varphi + \int_{\varphi_0'}^{\varphi_1'} J(\varphi)\sin\varphi\, d\varphi}{\int_{X_s}^{X_e} Xe(X)dX}, \qquad (47)$$

where: X_s, X_e are the coordinates of prescribed illuminance curve; φ_0, φ_1 are the limiting angles of reflector; φ_0', φ_1' are the boundary angles for direct light (see Fig.5).
Exclusion of critical points leads to narrowing of defined boundaries of illuminance curve, and hence to a change of scale multiplier *M*.
Use of flux balance equation (46) seems advantageous due to the following reasons:
(1) you may ignore the naught of prescribed indicatrix of a light source;
(2) calculation of profile can be reduced to solving of an algebraic equation together with one differential equation instead of two laborious differential equations.

Flux balance equation (46) defines RTF implicitly and generally can be solved by one of numerical methods applicable to non-linear equations (bisecting or Newton method, etc. [4]).

2.2.3. Approaches to solving equations of reflector profile curve

Differential equations like Eq.(45) may be solved by some numerical method, e.c. by Runge-Kutta method. Since multiplier *M* remains unknown, it may be found by iterating these equations. Use of a flux balance equation (46) enables to find *M* and to solve one differential equation. Here we have to set an acceptance angle φ_e of reflector. From the other hand we may find an acceptance angle with the help of Eq.(46) or by solving the system (45). These two approaches give the same final result [1].

2.2.4. Main equations for calculation of mirror shape

2.2.4.1. Translational symmetry (line light source)

Let neccessary to have illuminance distribution (ID) - E(x) on the target plane (TP). Than using equation of fluxes balance on entry and output of reflector we can write:

$$\int_{\varphi_s}^{\varphi} I(\varphi)d\varphi = \pm\rho^{-1}\left\{ M\int_{x_s}^{x} E(x)dx - \int_{x_s}^{x} I(\varphi_v)\left[1+x^2\right]^{-1/2} dx \right\} \qquad (48),$$

where

ρ - reflectance of mirror;
φ - polar angle of reflector point. φ is counted from OZ axis ("+" - clockwise);
φ_s - start angle of reflector;
x - coordinate on the target plane normalized to the height of light center;
φ_v - angle of direct light from source. $\varphi_v = \pi$ - arctg(x);

I(φ) - luminous intensity distribution of light source in the direction φ normalized to maximum - I_{max};
E(x) - illuminance distribution on the TP normalized to maximum - E_{max}.
Amplification factor:

$$M = \frac{E_{\max}}{(I_{\max}/h)}$$

Additional term in (48) takes into consideration direct light from source that must be substracted from given ID.

Passaging to beams with finite aperture we get:

$$M = \frac{\pm\rho\int_{\varphi_s}^{\varphi_e} I(\varphi)d\varphi + \int_{x_s}^{x_e} I(\varphi_v)\left[1+x^2\right]^{-1/2} dx}{\int_{x_s}^{x_e} E(x)dx} \qquad (49),$$

where

x_e - stop point of lit area on the TP;
φ_e - stop angle of reflector;

To provide solution it is neccessary to define border angles of reflector - φ_s, φ_e, borders of lit area on the TP - x_s,x_e and distributions - I(φ), E(x).

Equations (48) and (49) define dependence between incident rays and reflected rays. $\varphi = \varphi(x)$ ("ray tracing function").
When the dependence $\varphi(x)$ is defined, the shape of reflector can be found with the help of differential equation:

$$\frac{dr}{d\varphi} = rtg\frac{\varphi - \alpha}{2} \qquad (50),$$

where

$$\alpha = arctg\frac{r\sin\varphi - x}{r\cos\varphi + 1}$$ - polar angle of relfected ray corresponding to angle φ.

Thus shape of reflector will be described in polar form r = r(φ) and recount in Cartesian coordinates x = r sin(φ) and z = r cos(φ).

2.2.4.2. Rotational symmetry

By analogy with cylindrical symmetry we can write:

$$\int_{\varphi_s}^{\varphi} I(\varphi)\sin\varphi d\varphi = \pm\rho^{-1}\left\{ M\int_{x_s}^{x} xE(x)dx - \int_{x_s}^{x} xI(\varphi_v)\left[1+x^2\right]^{-3/2} dx \right\} \qquad (51)$$

Amplification factor:

$$M = \frac{E_{\max}}{\left(I_{\max}/h^2\right)},$$

or

$$M = \frac{\pm\rho\int_{\varphi_s}^{\varphi_e} I(\varphi)\sin\varphi d\varphi + \int_{x_s}^{x_e} xI(\varphi_v)\left[1+x^2\right]^{-3/2} dx}{\int_{x_s}^{x_e} xE(x)dx} \qquad (52)$$

where

$\varphi_v = \pi$ - arctg(x);

2.2.4.3. Translational symmetry (point light source)

Common formula for calculation of horizontal illuminance on the target plane displaced from point [0,0,0] with light source on the height h is:

$$E = \frac{\rho I(\varphi,\beta)\sin\varphi}{D(\varphi,\beta)},$$

where

$D(\varphi,\beta)$ - Jacobian of coordinate conersion φ,β -> x,y;
ρ - reflectance;
φ,β - spherical coordinate of ray incident on the reflector;
x,y - coordinate of ray on the plane XOY;
$I(\varphi,\beta)$ - luminous intensity of light source in the direction φ,β

For the main plane (beta = 0) we get:

$$E(x,y) = \frac{\rho I(\varphi,\beta=0)\sin\varphi}{D(\varphi,\beta=0)}$$

Calculate partial derivatives:

$$\frac{dx}{d\beta} = 0$$

$$\left.\frac{dx}{d\varphi}\right|_{\beta=0} = \frac{dx}{d\varphi}$$

$$\frac{dy}{d\beta} = \sin\varphi\left\{r + \frac{h + r\cos\varphi}{\cos\alpha}\right\},$$

where

α - angle of reflected ray to axis OZ.

We get:

$$D = \left|\begin{matrix} \frac{dx}{d\varphi} & \frac{x}{d\beta} \\ \frac{dy}{d\varphi} & \frac{dy}{d\beta} \end{matrix}\right|_{\beta=0} = \frac{dx}{d\varphi}\sin\varphi\left\{r + \frac{h + r\cos\varphi}{\cos\alpha}\right\}$$

Thus:

$$E = \rho I(\varphi)\frac{d\varphi}{dx}\frac{1}{r + \dfrac{h + r\cos\varphi}{\cos\alpha}}$$

We get system of differential equations for calculation of mirror profile (for plane $\beta = 0$):

$$\begin{cases} \dfrac{d\varphi}{dx} = \dfrac{rE(x)}{\rho I(\varphi)}\left\{1 + \dfrac{h/r + \cos\varphi}{\cos\alpha}\right\} \\ \dfrac{dr}{dx} = rtg\left(\dfrac{\varphi - \alpha}{2}\right) \end{cases} \qquad (53)$$

where

$$tg\alpha = \frac{r\sin\varphi - x}{r\cos\varphi + h}$$

This system can be solved with the help of Runge-Kutta's method.

2.2.4.4. Choice of sign

As left parts in the formulas (49) and (52) are positive on principle sign ("+" or "-") of right parts must be chosen from this condition.Thus "+" is chosen if sign of angle change from φ_s to φ_e coincides with the sign of coordinate change from x_s to x_e. Otherwise "-" is chosen.

2.2.4.5. Critical points

These are points (x_{cr}) where expressions in the right parts of equations (48) and (51) are equal to zero.

Thus for the translational problem:

$$E(x_{cr}) = M^{-1} I\left(\pi - arctg(x_{cr})\right)\left(1 + x_{cr}^2\right)^{-1} \qquad \text{(a),}$$

for the rotational problem:

$$E(x_{cr}) = M^{-1} I\left(\pi - arctg(x_{cr})\right) x_{cr}\left(1 + x_{cr}^2\right)^{-3/2} \qquad \text{(b),}$$

In this case light aperture of reflector is wider than it is neccessary and so dimensions of lit area must be decreased.

As appears from (a) and (b) the more given amplification factor, the less will be the difference between given distribution curve and turned out curve.

2.2.4.6. Duality of solution method

The problem of calulating the mirror shape has an alternative approach: define start angle of reflector - φ_s, distribution - I(φ), distribution - E(x) in the range [x_s,x_e] and amplification factor. Final angle of reflector can be found in the proccess of calculation.

Thus the values φ_e and M are interchangeable. This means duality of problem solution.

2.2.4.7. Normalization of distributions to flux 1000 lm

Let flux in the range φ_s,φ_e:

$$F(\varphi_s, \varphi_e) = \begin{cases} \int\limits_{\varphi_s}^{\varphi_e} I(\varphi) d\varphi, & translaional \quad symmetry \\ 2\pi \int\limits_{\varphi_s}^{\varphi_e} I(\varphi) \sin\varphi d\varphi, & rotational \quad symmetry \end{cases}$$

where

I(φ) - curve in the relative units.

Than total flux from the light source:

$$F_t = \begin{cases} F(0,2\pi), & translational \quad symmetry \\ 2\pi F(0,\pi), & rot\,ational \quad symmetry \end{cases}$$

Scale multiplier for LID of light source:

$$A = 1000/F_t$$

Scale multiplier for flux from ID is equal:

$$A' = \frac{A(F_{refl} + F_{dir})}{F_{id}},$$

where

F_{refl} - reflected flux;
F_{dir} - flux on the TP from light source;
F_{id} - flux from given ID.

Thus we get:

$$A' = M\,A$$

Scale multiplier for recounting of ID in the absolute values:

***rotational symmetry*:**

$$A'F_{id} = \int Xe(X)dX = mh^2 \int xE(x)dx\,,$$

where

X - absolute values of coordinates;
e(X) - absolute function values of ID.

Thus

$$m = A'/\,h^2$$

translational symmetry:

$$m = A'/h$$

2.2.4.8. Calculation of utilization factor

Utilization factor:

$$K_{util} = \frac{A'F_{id}}{1000}$$

2.2.4.9. Method of solution

Equations (48), (49) and (51), (52) can be solved with one of numerical method of equation solution (method of bisection, Newton, secants and so on).

Algorithm of function Zeroin [4] is quite acceptable for this solution.

Numerical solution of equation (50) is realized with the help of Runge-Kutta's method of 4-th order with automatic choice of integration step. Runge-Kutta's method in comparison with others methods of prediction-correction doesn't require calculation of start function values. And calculation of function $\varphi = \varphi(x)$ incoming in right part of equation (50) reduces to excerption of values with help of spline-interpolation.

2.2.4.10. Integration of light distribution function

The input distribution are presented as a set of points so the choice of intermediate value can be realized with the help of linear or spline interpolation. It means that the flux distribution can be found analyticaly if the type of interpolation between knots is known.

Let $x_1 = x[i]$ and $x_2 = x[i\text{-}1]$, where i - the number of interpolation knot. Function values: $y_1 = f(x_1)$ and $y_2 = f(x_2)$. And $u = x - x_1$, where $x_1 < x < x_2$. Step: $h = x_2 - x_1$.

(1) The linear interpolation case

Translational symmetry

Function of flux increasing from light distribution (LID or ID).

$$dF = u\{y_1 + 0.5u(y_2 - y_1)/h\}$$

Rotational symmetry

Flux increasing function for LID of light source.

$$dF = \{y_1[h\cos(x_1) - (x_2 - u)\cos(u)] - y_2(u - x_1)\cos(u) + (y_2 - y_1)(\sin(u) - \sin(x_1))\}/h$$

Flux increasing function for ID on the target plane.

$$dF = (1/6)(u - x_1)\{y_1(2x_1 + u) + y_2(2u + x_1)\}$$

(2) The spline interpolation case

$$y = y_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3$$

where

b_1, c_1, d_1 - spline interpolation coefficients.

Translational symmetry

Function of flux increasing from light distribution (LID or ID).

$$dF = u\left\{y_1 + u\left[(1/2)b_1 + u\left((1/3)c_1 + u(1/4)d_1\right)\right]\right\}$$

Rotational symmetry

Flux increasing function for LID of light source.

$$dF = y_1 I_{\sin} + b_1(I_{x\sin} - x_1 I_{\sin}) + c_1(I_{xx\sin} - 2x_1 I_{x\sin} + x_1^2 I_{\sin}) + d_1(I_{xxx\sin} - 3x_1 I_{xx\sin} + 3x_1^2 I_{x\sin} - x_1^3 I_{\sin})$$

where

$I_{sin} = U_c$
$I_{xsin} = U_s + V_c$
$I_{xxsin} = W_c + 2\cdot(V_s - U_c)$
$I_{xxxsin} = T_c + 3\cdot W_s - 6\cdot(U_s + V_c)$
$U_c = \cos(x_1) - \cos(x)$
$U_s = \sin(x) - \sin(x_1)$
$V_c = x_1\cdot\cos(x_1) - x\cdot\cos(x)$
$V_s = x\cdot\sin(x) - x_1\cdot\sin(x_1)$
$W_c = x_1^2\cos(x_1) - x^2\cos(x)$
$W_s = x^2\sin(x) - x_1^2\sin(x_1)$
$T_c = x_1^3\cos(x_1) - x^3\cos(x)$

Flux increasing function for LID of light source.

$$dF = u\left[\left((1/2)u + x_1\right)y_1 + u\left[\begin{matrix}\left((1/3)u + (1/2)x_1\right)b_1 + \\ u\left[\begin{matrix}\left((1/4)u + (1/3)x_1\right)c_1 + \\ u\left((1/5)u + (1/4)x_1\right)d_1\end{matrix}\right]\end{matrix}\right]\right]$$

The above mentioned formulas are used for flux calculation at the interpolation knots when $u = h$.

If F_1 - the flux value at the knot $x_1 = x[i]$, then returning value at the point x:

$$F(x) = F_1 + dF$$

3. Program realisation

3.1. Peculiarities of numerical calculations

3.1.1. Direct calculations

Stipulation of differentiation process

Calculation of illuminance is carried out with the aid of Eqs.(8)-(26). Expressions for luminous intensity can be easily obtained from the equations for illuminance. Presence of focal points can be discovered if we set naughts in denominators of appropriate expressions.
Profile curve of reflector may be specified either in the form of $r(\varphi)$ or $z(X)$, or with the aid of RTF. Since Eqs.(8)-(26) demand calculation of $d\alpha / d\varphi$, a method of interpolation will be important. Representation by cubic spline can be suitable [6].
It is obvious from Eqs. (27), (29) that double differentiation is needed under coordinate prescription of a profile. Differentiation is poorly stipulated operation [7] with condition number $v=2/h$, where h is a table step. If an error of numerical differentiation $r_d = 1/2h$ is decreasing under $h \to 0$ then $v \to \infty$. Hence, oscillations appear in illuminance curve with a frequency of nodes selection, and their amplitude sharply increases with reduction of step.

Smoothing of curves

We smooth over equally spaced nodes f_k with the aid of a linear expression [8]

$$g_k = (1/3)*(f_k + f_{k+1} + f_{k+2}) \qquad (54)$$
$$X_k = h*k.$$

Let $f(x) = e^{2\pi*i*s*x}$ be a simple sinusoid that is given to the entrance of smoothing filter (54). Let $h=1$, and $2\pi s = \omega$, then $f_k = e^{i\omega k}$, therefore we obtain

$$g_k = \frac{e^{i\omega(k+1)}}{3}(e^{-i\omega} + 1 + e^{-i\omega}).$$

So, the values f are being multiplied by χ

$$\chi = \left| \frac{1 + 2\cos\omega}{3} \right| . \qquad (55)$$

The latter is equal to null if $\omega = 2\pi/3$ or $s=1/3$, i.e. the smoothing formula creates a strong suppression effect in the upper part of frequency spectrum (Fig.5). Further smoothing by four nodes makes even greater effect

$$g_k = (1/4)*(f_k + f_{k+1} + f_{k+2} + f_{k+3} \quad (X_k = hk) \qquad (56)$$

with a spectrum

$$\chi = \left| \frac{\sin 2\omega}{4\sin(\omega/2)} \right|$$

that has naughts under frequency *s=1/4, 1/2, ...* (see Fig.6) .

3.1.2. Inverse calculations

When computing reflector profile a ray-tracing function $\varphi(x)$ is determined by numerical solution of (non-linear) equation of flux balance (Fig. 6). Bisecting method is the most valid, but at the same time it is the slowest. Procedure ZEROIN, which combines validity of bisecting method and speed of parabolic interpolation method, proved to be good for our purposes [3]. Newton's methods do not suit well for this purpose, since they need computations of derivatives.
When solving Cauchy problem we use Runge-Kutta procedure of the fourth order with automatic selection of step [4]. The reason is that multistep methods (i.e., Adams-Bashfort method) need several starting points. Besides, computation of the right part of Eq.(45) is rather easy.

3.2. Distinctive peculiarities

The main peculiarities of software are as follows:
- only terms of geometrical optics ("a thin geometrical ray") are used in calculation of reflectors, that provides high accuracy of results;
- unrestricted number of light sources can be used in design (for direct problem);
- position of a light source is arbitrary, that enables to account a defocusing of a system;
- utilisation factor is calculated;
- profile curve of reflector can be specified in different forms: Cartesian representation, polar or with the aid of ray-tracing function;
- since the direct problem is solvable for arbitrary ray-tracing function, aberrations of reflector can be taken into account;
- to compute only direct illumination is possible.

3.3. Input data

Input data are divided on several sections:
- system and problem type;
- light source;
- reflector;
- illuminated area.

3.3.1. System and problem type

The following parameters are selected as initial to define optical system:
- problem type (direct, inverse);
- system symmetry type (translational, rotational);
- light spread type (illuminance, luminous intensity);
- accounting of direct light of a source.

3.3.2. Light source

The input parameters of a light source are as follows:

- luminous intensity curve (in relative units)

$$I(\varphi) = \begin{cases} -\pi \le \varphi \le \pi, & translational \quad symmetry \\ \hline 0 \le \varphi \le \pi, & rotational \quad symmetry \end{cases},$$

  and its interpolation type: linear or cubic spline. Under linear interpolation it is possible to define light spread function by two points;
- position of a light source.

3.3.3. Reflector

In case of direct problem reflector can be defined in different forms: Cartesian representation, polar or with the aid of ray-tracing function. In the latter case, initial radius has to be specified. If a task of profile design is put forward than the following parameters have to be defined:

- initial polar angle and corresponding radius-vector;
- final polar angle;
- number of calculated points on a profile.

3.3.4. Illuminated area

In case of direct problem illuminated area is specified by boundary points: coordinates on a task plane (in case of calculating illuminance) or by angles α (in case of calculating luminous intensity). When calculating illuminance curve, a distance to a task plane has to be specified. The distance may be negative; the latter case means that a task plane is placed above a light source, so a direct light is ignored. Number of points on calculated light spread curve must be prescribed as well.

In case of inverse problem a light spread curve must be specified, as well as interpolation type and boundaries of illuminated area. Under linear interpolation step-functions can be represented easily, i.e. uniform light spread that is important for practice.

4. Test example for controlling program operation

4.1. Direct problem

We calculate distribution of illuminance produced by reflector having translational symmetry. Reflector is specified by elliptical profile described by the following polar equation (Fig. 7)

$$r(\varphi) = 9000 / (5 + 4\cos\varphi).$$

A light source has a uniform distribution of luminous intensity and is placed in the first focus of ellipse.

Table 3

Calculated and theoretical profiles of reflector

N	φ, deg.	R, mm	R_{ell}, mm	dR/R, %
1	25.00	6546.5551	6546.5551	-0.0000
2	30.00	5860.3377	5859.7627	0.0098
3	35.00	522.5607	5222.2599	0.0058
4	40.00	4649.4830	4649.1872	0.0064
5	45.00	4144.6936	4144.4614	0.0056
6	50.00	3705.6536	3705.4580	0.0053
7	55.00	3326.4812	3326.3182	0.0049
8	60.00	3000.1379	3000.0000	0.0046
9	65.00	2719.5396	2719.4219	0.0043
10	70.00	2478.1303	2478.0285	0.0041
11	75.00	2270.1082	2270.0194	0.0039
12	80.00	2090.4728	2090.3946	0.0037
13	85.00	1934.9805	1934.9109	0.0036
14	90.00	1800.0624	1800.0000	0.0035
15	95.00	1682.7325	1682.6761	0.0034
16	100.00	1580.4979	1580.4467	0.0032
17	105.00	1491.2793	1491.2325	0.0031
18	110.00	1413.3416	1413.2987	0.0030
19	115.00	1345.2357	1345.1964	0.0029
20	120.00	1285.7505	1285.7143	0.0028
21	125.00	1233.8725	1233.8391	0.0027
22	130.00	1188.7536	1188.7229	0.0026
23	135.00	1149.6845	1149.6562	0.0025
24	140.00	1116.0727	1116.0468	0.0023
25	145.00	1087.4262	1087.4020	0.0022

Illuminated plane is situated at a distance of 20 m from system focus. The boundaries of illuminated area are: from -16063.6 mm to -841.83 mm.

4.2. Inverse problem

Under set distribution of illuminance shown in Fig.2 calculation of reflector profile has to be carried out.
The initial parameters of reflector: $\varphi_s = 155^0$, $r = 6546.56\,mm$, $\varphi_e = 35^0$ (that corresponds with direct problem 4.1).
The results of calculation are given in Table 3. There are polar coordinates of initial (elliptic) profile that are given for comparison. Maximal related error does not exceed 0.01%.

5. Direct and inverse problems in case of extended sources

In case of designing reflectors operating with extended sources the concept of integrating both direct and inverse problems in a single package remains unchanged. The common features of calculations are as follows:

(1) A function of axial rays traces or ray-tracing function (RTF) serves as a main characteristic in determining flashed area and light spread of specular reflector;

(2) There is a choice from two ray-tracing schemes (or from two types of reflectors), that is, from direct and crossing;

(3) Solution of inverse problem is being constructed on solution of direct problem where light spread is determined under specified characteristics of a light source and reflector geometry;

(4) In inverse calculations with extended sources we totally use the relation between geometrical properties of conic sections and concentrating properties of reflectors being synthesised from said curves;

(5) When operating with several sources, a direct problem can be solved as well

The peculiarities of solving problems with extended sources:

(1) calculations are being made by luminance and image area, i.e. Mangene law is applicable. That leads to elimination of infinities in light spread curves connected with δ-function of light source luminance;
(2) a function of angular dimensions of a light source (elementary map) is introduced that together with RTF defines a flashed area of reflector [1,3];
(3) precise analytical equations exist that enable to solve direct problem with a point source; generally, dimensions of flashed area that define intensity of illumination can be found only with the aid of iteration methods (inverse-ray method, elementary map method, etc.);
(4) it is necessary to calculate an own light spread of a light source, while in case of a point source this characteristic was initial;
(5) it is necessary to take into account a shadowing of reflector flashed area, as well as a mutual shadowing between sources with finite dimensions;
(6) the inverse problem in case of extended sources, unlike for point sources, belongs to incorrect problems similar to Fredholm equation of the 1st-kind; so, existence, uniqueness, and steadiness of solution are very important questions here (while in the case of a point source a profile curve can be found practically for any specified luminous intensity distribution,

in case of extended source the solution does not exist if light spread curve has sharp shifts in its profile);

(7) inverse problem is being solved in the process of dealing with several direct problems, i.e. in iteration cycle and adjustment with prescribed light spread curve; that takes much more time than for solving problems with a point source.

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APPENDIX.

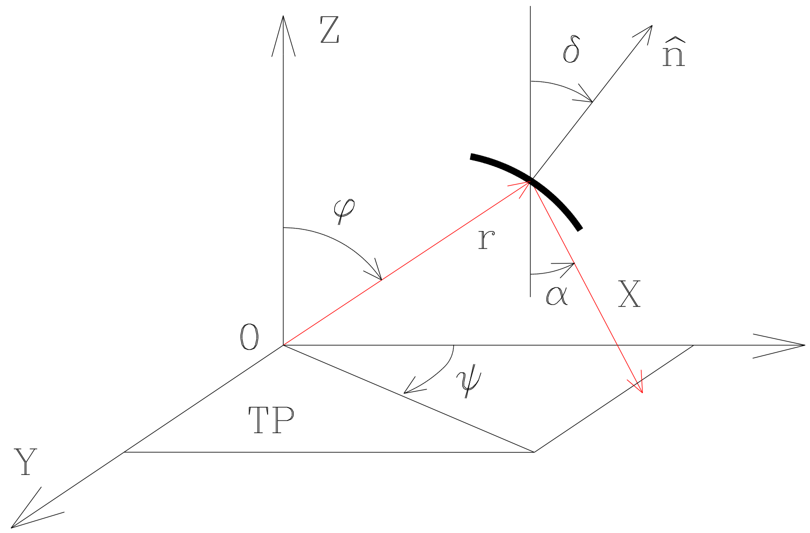

Fig. 1. To deduction of formulas for calculating intensity.

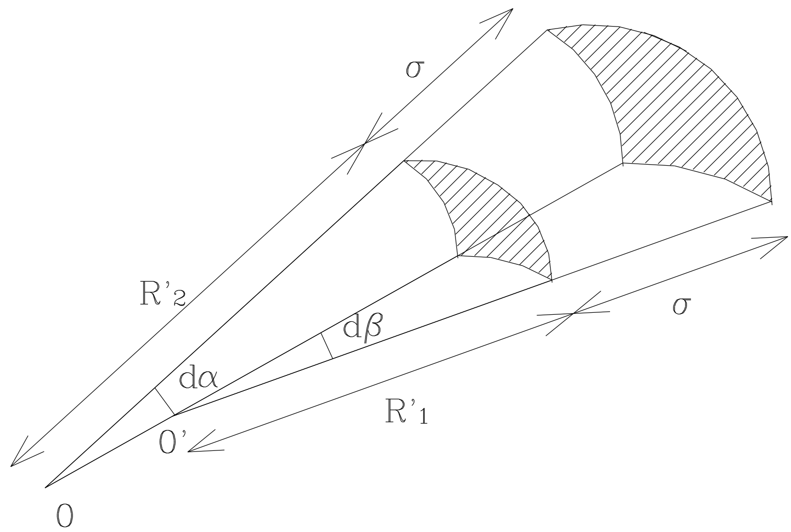

Fig. 2. Wave fronts in astigmatic tube.

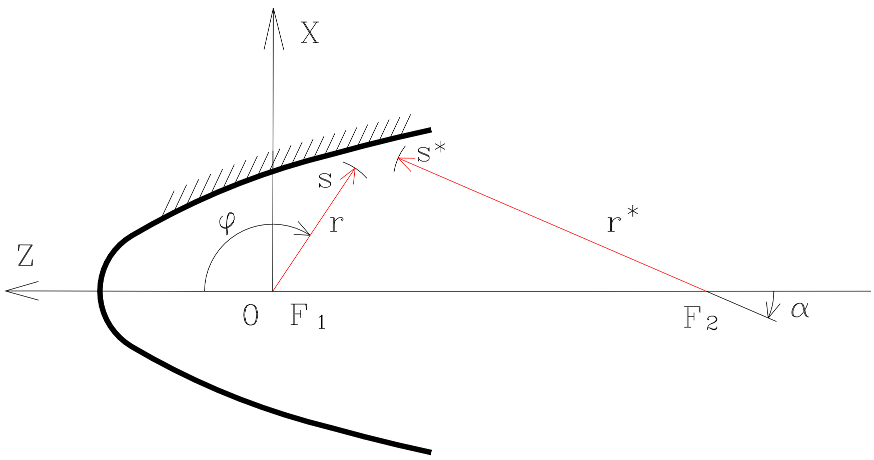

Fig. 3. To deduction of equation for ellipsoid luminous intensity.

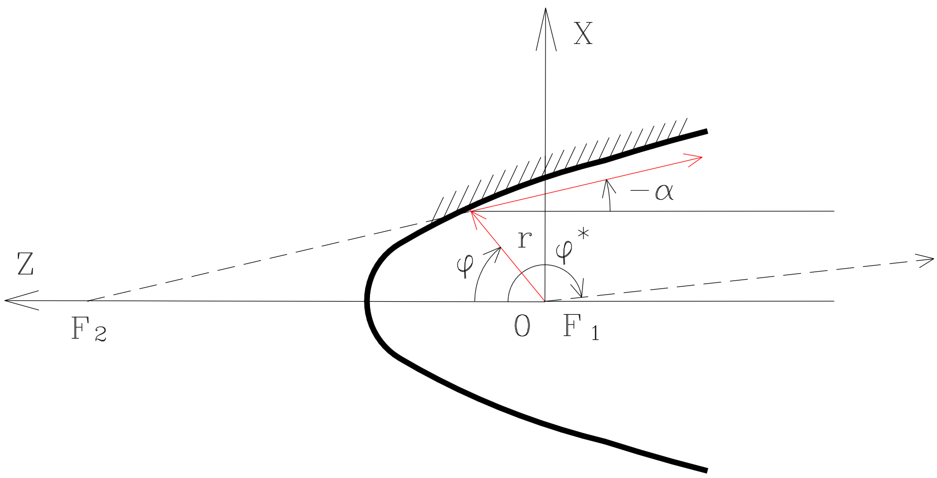

Fig. 4a. Variations of efficient angle j for concave branch of hyperbola.

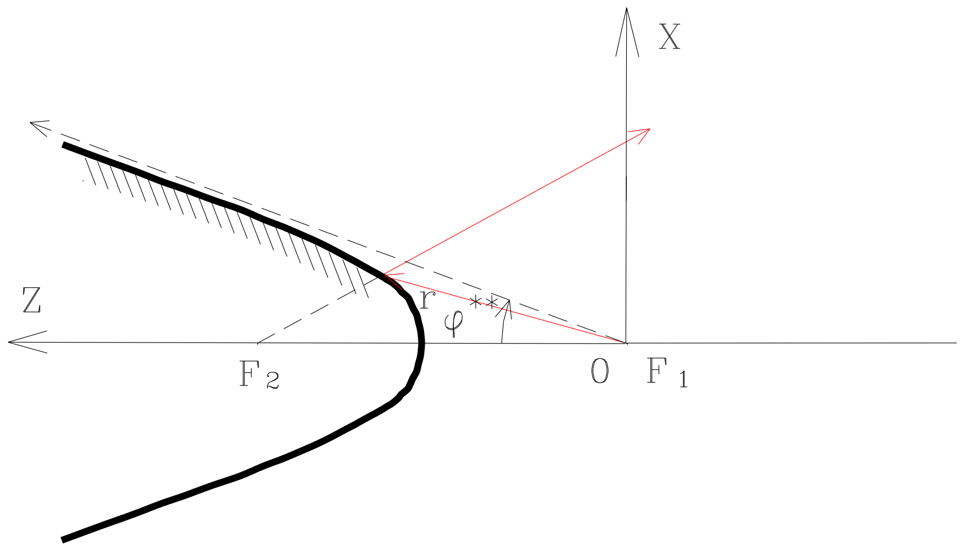

Fig. 4b. Variations of efficient angle j for convex branch of hyperbola.

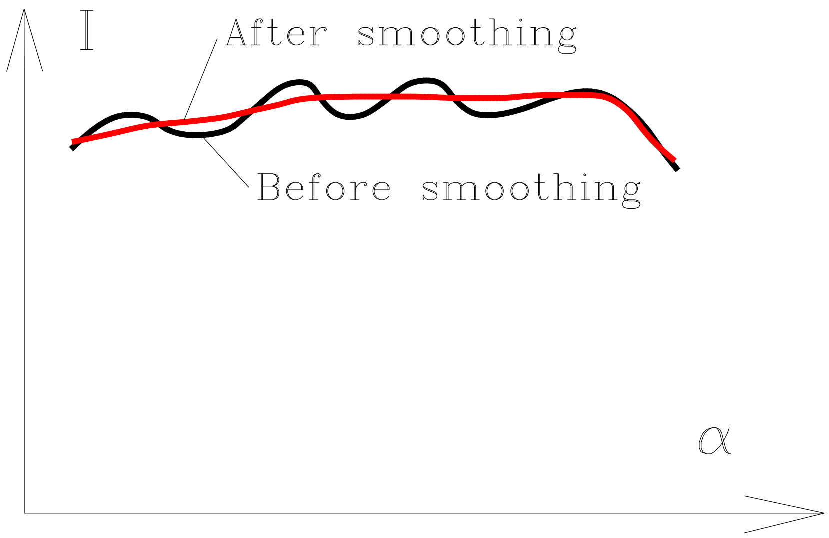

Fig. 5. Smoothing of luminous intensity curve for reflector assigned by points.

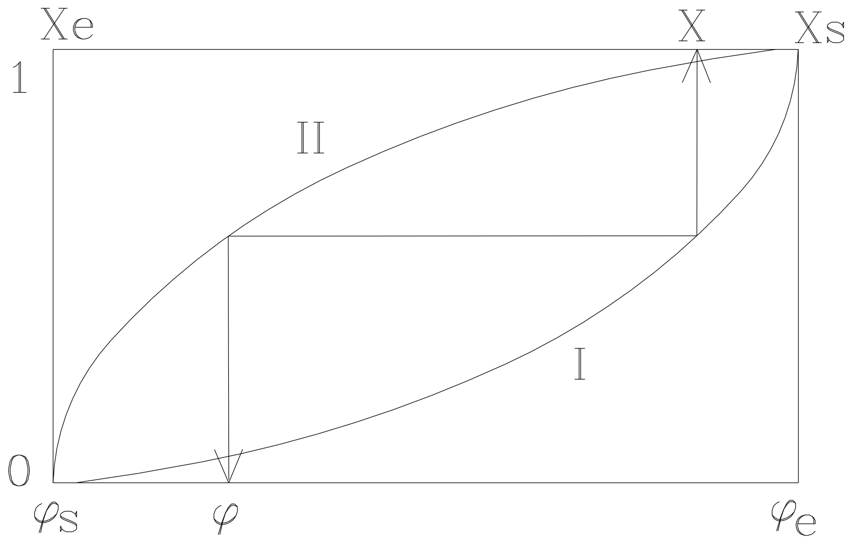

Fig. 6. Determination of RTF from partial flux balance equation.